Derived Geometry and Physics

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Abstract

This series of lectures, which essentially consists of three parts, serves a conceptional introduction to the interaction between derived geometry and physics based on the formalism that has been heavily studied by Kevin Costello. Main motivations of our current lecture series are as follows: *(i)* to provide a brief introduction to *derived algebraic geometry*, which can be, roughly speaking, thought of as a higher categorical refinement of an ordinary algebraic geometry, *(ii)* to understand how certain *derived objects* naturally appear in a theory describing a particular physical phenomenon and give rise to a formal mathematical treatment, such as redefining a perturbative classical field theory (or its quantum counterpart) by using the language of *derived algebraic geometry*, and *(iii)* how the notion of *factorization algebra* together with certain higher categorical structures come into play to encode the structure of so-called *observables* in those theories by employing certain cohomological/homotopical methods. Adopting such a heavy and relatively enriched language allows us to formalize the notion of *quantization* and observables in a quantum field theory as well.

In the first part of the series, which can be considered as a warm-up tour, to motivate the underlying mathematical treatment encoding the *derived set-up* we shall focus on the notion of *stack* which can be thought of as a first instance such that the ordinary notion of category *no longer* suffices to define such an object. To make sense of this new object in a well-established manner and enjoy the richness of this *new structure*, we need to introduce a higher categorical notion, namely a *2-category*. Adopting this higher categorical formalism, we shall re-visit so-called *the moduli problem (manifestly given by a certain functor)* and study on how introducing *the 2-category of stacks* ensures the representibility of this functor which, in general, fails to be representable in the category of S-schemes. In other words, stacks and 2-categories serve as a motivating/prototype conceptual examples before introducing the notions like ∞-*categories, derived schemes, higher stacks and derived stacks.*

In the second part of the series, we shall generalize and enlarge the formalism that we have already employed in the first part of the series. We shall introduce the notion of *derived schemes* by adopting *the functor of points*-type approach in a way that one can rather easily realize the essential difference between derived and underived settings. In accordance with this approach, we analyze the local model for a derived k-scheme $(X, \mathcal{O}_X)$, which turns out that the structure sheaf $\mathcal{O}_X$ is a sheaf of commutative differential graded algebras when $char(k) = 0$. Again the notion of category is still too naïve to capture the derived nature, and hence one has to introduce the ∞-*category* of derived schemes. However, how to construct an ∞-category is an another story *per se* and rather complicated. A relatively reasonable way is by so-called *model categories.* Therefore, we shall only interested in ∞-categories arising from certain model categories. To make the first touch with physics and realize where derived geometry comes into play, we shall discuss further notions and structures in a rather intuitive manner, such as *derived critical locus* and *the symplectic structure* on this derived object.

In the last part of the series, we shall investigate (i) the *derived* interpretation of a field theory in a sense that employing the Lagrangian formalism, for instance, one can realize a classical field theory as *the derived critical locus* of the action functional since it can be described as a *formal moduli problem* cut out by a system of PDEs determined by the corresponding Euler-Lagrange equations governing the system under consideration, and (ii) Costello's approach to the structure of observables in a perturbative quantum field theory in the language of a conceptual framework so-called *factorization algebra* together with cohomological methods naturally arise in the theory of derived algebraic geometry. To be more specific, Costello's main motivation to study factorization algebras associated to a *perturbative* QFT is to generalize the deformation quantization approach to quantum mechanics developed by Kontsevich. In other words, deformation quantization essentially encodes the nature of observables in *one*-dimensional quantum field theories and factorization algebra formalism provides a n-dimensional generalization of this approach.