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Abstract

In this paper we are going to give a short introduction to Seiberg-Witten duality in N=2 Super-Yang-Mills theory without matter fields and for SU(2) gauge group. The material is heavily drawn from Seiberg and Witten's original article. An elementary account of monopole solutions in gauge theories is added. $N = 1$ and $N = 2$ supersymmetries are introduced. Main analysis is focused on monopole condensation and confinement in $N = 1$ theory after softly breaking $N = 2$.

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The success physicists had in describing the Electroweak and Strong Interactions almost proves that Quantum Field Theories are the correct ways of understanding the Nature. But in all physical theories our understanding has been mostly perturbative and more embarrassingly the best we can do is to calculate the first several terms in the expansions. This does not generate a big problem for weakly interacting theories like QED (at not very large energies). But asymptotically free theories exhibit some phenomena such as confinement and spontaneous breaking of chiral symmetry which can only be understood by means of non-perturbative methods. There have been a great deal of developments in non-perturbative physics since the first appearance of Polyakov's paper on instantons. But still in realistic theories, like QCD, we have little control of calculations in the strongly interaction regime.

The last decade brought a lot of excitement about non-perturbative physics in strongly interacting non-Abelian gauge theories. Since the discovery of N=1 supersymmetry it is well known that SUSY field theories are perturbatively much easier to handle. As it turned out the more supersymmetry we put in the more control we have not only on the perturbative regimes but also on the non-perturbative regimes as well. Some very interesting exact statements were made in the 80s. Just to mention a couple examples; N= 4 SUSY was shown to be finite at every order in perturbation theory and does not have any non-perturbative corrections and by using the holomorphicity arguments Shifman et.al were able to obtain the exact perturbative beta functions in some SUSY theories. In the past years Seiberg was able to extract a lot of information about the non-perturbative regimes in different theories and specifically N=2 Super-Yang-Mills. And in 1994 Seiberg and Witten by using holomorphicity and a version of electromagnetic duality were able to solve **N=2 SUSY QCD** exactly. To be specific they found the exact masses of particles (vector bosons and dyons) and determined the metric on the moduli space of the vacua. Their solution essentially depends on reformulation of a strongly interacting non-Abelian theory of electrically charged particles in terms an of Abelian theory of weakly interacting magnetic monopoles and dyons. By using the Dual Meissner effect (condensation of monopoles) they were able to show how confinement of electric charge takes place. This solution not only was the first ever example of an exactly soluble four dimensional non-trivial field theory but it also opened an era of dualities in physics and mathematics. String theorists enjoyed and still enjoying this very powerful idea and they are trying to unify the existing five different string theories. Witten quickly realized that Seiberg-Witten duality is a powerful tool in the classification of four manifolds. And his paper on "Monopoles and Four Manifolds" was very welcome in the mathematics community. I will briefly touch in this very interesting issue. As I have stated SW solution crucially depends on three ingredients supersymmetry, holomorphy and duality which are to be discussed in the following.

DIRAC'S MONOPOLE AND ELECTRIC-MAGNETIC DUALITY

Electric-magnetic duality has been around probably since Maxwell. In vacuum one can write the Maxwell's equations in the following form.

$$\nabla \cdot (\vec{E} + i\vec{B}) = 0, \qquad \nabla \wedge (\vec{E} + i\vec{B}) = i\partial_t(\vec{E} + i\vec{B}) \tag{1}$$

In addition to conformal symmetry these equations display an other global symmetry which is $\vec{E}+i\vec{B} \rightarrow e^{i\phi}(\vec{E}+i\vec{B})$. One observes that this symmetry is only valid in four dimensions where both electric and magnetic fields are vector quantities. Energy and momentum density are invariant under this transformation but the Lagrangian forms a doublet with the topological quantity $i\vec{E} \cdot \vec{B}$. Dirac extended this trivial-looking duality by including matter. It is clear that one has to introduce magnetic charge and magnetic current to extend the duality. The equations become

$$\nabla \cdot (\vec{E} + i\vec{B}) = \rho_e + i\rho_m, \quad \nabla \wedge (\vec{E} + i\vec{B}) = i\partial_t(\vec{E} + i\vec{B}) + i(j_e + ij_m) \tag{2}$$

Where (ρ_e, j_e) and (ρ_m, j_m) correspond to electric charge density, current and magnetic charge density and current respectively. As far as classical electrodynamics is concerned we really have not done anything new but to assume that there are monopoles. But the requirement that this theory be consistent quantum mechanically has far more reaching consequences. To begin with one has to introduce a vector potential in the quantum theory which is the only way to couple charged particle to electromagnetic fields. But if a vector potential is introduced magnetic field is given by the relation $\vec{B} = \vec{\nabla} \wedge \vec{A}$. This gives a divergence free magnetic field as a vector identity. So it looks like quantum theory does not allow a magnetic monopole unless one has a singularity in the vector potential. Following Dirac we introduce a semi-infinite solenoid attached to a magnetic monopole. Suppose the monopole is fixed at the origin and by rotational symmetry we choose the solenoid (string) to extend along the negative z axis. Then the magnetic field is given by

$$\vec{B} = \frac{g\vec{e}_r}{4\pi r^2} + g\theta(-z)\delta(x)\delta(y)\vec{e}_z \tag{3}$$

It is clear that this string is an artificial construction. Its existence should not effect the physics. This leads to the non-existence of Aharonov-Bohm effect for electrically charged particles that pass by the sides of the string. (Or in other words wave function of a charged particle should not pick up an observable phase after a rotation of 2π around the string) And this suggests

$$e\oint \vec{A}_{string} \cdot d\vec{l} = e\int_S \vec{B} \cdot d\vec{S} = eg \tag{4}$$

This phase should be equal to $2\pi n$. So one obtains the celebrated Dirac's quantization condition: $eg = 2n\pi$.

The occurrence of this string singularity makes it hard to develop a quantum field theory for monopoles. Although above analysis looks rather trivial it actually is quite interesting. As Dirac suggested this might be the explanation of electric charge quantization if there exits at least one monopole in the universe.

Dirac's quantization condition breaks the above mentioned rotation of the electric magnetic fields into a discrete one, i.e $\vec{E} \to \vec{B}, \vec{B} \to -\vec{E}$. This is because Dirac did not allow magnetically charged objects to have electric field. Zwanziger and Schwinger cured this problem and introduced the concept of a "**dyon**". The quantization condition becomes , for two dyons (g_1, q_1) and (g_2, q_2) : $q_1 g_2 - q_2 g_1 = 2\pi n\hbar$. Dirac's construction does not depend on any specific theory and so there is nothing more to say about what a monopole is except to look it experimentally. Over the years the non-observation of the monopoles was attributed to their mass being very large. One actually wants to go further than what Dirac did and find a theory where one can calculate the spin and mass of the monopole and determine the structure of it.

A better understanding of the monopole came with the works of 't Hooft and Polyakov. They observed that to have a monopole solution the essential ingredient is to have a compact $U(1)$ group in the theory. If the gauge group is compact and not simply connected there can be gauge transformations with nontrivial winding number. or in short fundamental group will be nontrivial. In Dirac's construction introduction of the monopole by hand compactifies the familiar $U(1)$ group. And the fundamental group ,$\Pi_1(U(1)) = Z$. Georgi-Glashow model with a gauge group $SU(2)$ or $SO(3)$ is the perfect example to look for monopoles. Since these groups are compact one obtains a compact $U(1)$ subgroup after spontaneous symmetry breaking. I would like to dwell on this model a little more because not only this model has monopole solutions but also at the semi-classical level (and BPS limit) it supports the idea of exact electromagnetic duality. I say semi classical because as we will see quantum corrections alter the tree level equations and ruin the duality in any non-supersymmetric theories like this one.

't HOOFT-POLYAKOV MONOPOLE, GEORGI-GLASHOW MODEL AND MONTONEN-OLIVE DUALITY

We are interested in the Yang-Mills-Higgs sector of the Georgi-Glashow theory. The model is defined by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D_\mu\Phi)^a(D_\mu\Phi)_a - \frac{\lambda}{4}(\Phi^a\Phi^a - v^2)^2 \tag{5}$$

The Latin indices are gauge group indices which we take to be SO(3). We put the Higgs field in the adjoint representation. $F^a_{\mu\nu}$ is the Yang-Mills field strength and D^a_μ is the covariant derivative. This Lagrangian defines a renormalizable field theory and one can find the particle spectrum by looking at the perturbative excitations from the vacuum. And if the Higgs field develops a non vanishing value, i.e v non-zero , then the perturbative spectrum in a convenient gauge consists of two charged massive gauge bosons , a massless photon and a spin zero massive scalar field. In addition to these one can also show that there are topologically non-trivial soliton solutions with finite energy which could be identified as extended particles. Depending on the assumptions these solutions can carry magnetic charge and/or electric charge. Although explicit solutions for both monopoles and dyons can be constructed we will adopt a different way. We first assume that there exits a finite energy solution and we will show that it carries a magnetic charge. One might be puzzled with the role of the Higgs field. But as it will be clear in a moment the magnetic charge of the fat (non-elementary excitation) particle purely comes from the Higgs field. The Higgs field Φ in the vacuum satisfies the following equations,

$$\Phi^a_{vac}\Phi^a_{vac} = v^2, \qquad \partial_\mu\vec{\Phi} - e\vec{W}_\mu \wedge \vec{\Phi} = 0 \tag{6}$$

The most general solution for $\vec{W}_\mu$ is given by

$$\vec{W}_\mu = \frac{1}{ev^2}\vec{\Phi} \wedge \partial_\mu\vec{\Phi} + \frac{1}{v}\vec{\Phi}A_\mu \tag{7}$$

The first term is a particular solution and the second term is the general solution for the homogeneous equation where A_μ is arbitrary. By using the above equation one can find the non-Abelian field strength and then one defines an Abelian field which corresponds to the unbroken $U(1)$ gauge in the vacuum. So the Abelian field strength is

$$F_{\mu\nu} = \frac{1}{v}\Phi^a F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{v^3 e}\vec{\Phi}\cdot(\partial_\mu\vec{\Phi} \wedge \partial_\nu\vec{\Phi}) \tag{8}$$

It is straight forward to show that this tensor satisfies the Maxwell's equations by using the Higgs vacuum equations. The magnetic filed is given by $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$. So the monopole charge will be

$$g = \int\limits_\Sigma B^i dS^i = -\frac{1}{2ev^3}\int\limits_\Sigma \epsilon_{ijk}\vec{\Phi}_{vac}\cdot(\partial^j\vec{\Phi}_{vac} \wedge \partial^k\vec{\Phi}_{vac})dS^i \tag{9}$$

The first term in the Abelian field strength does not contribute due to the stokes theorem. (Clearly we are integrating over a compact surface which does not have a boundary) The last expression is a topological quantity related to the the winding number. We can understand it as follows; the vacuum of the theory ,$V(\Phi) = 0$, is a two-sphere so the Higgs field Φ in the vacuum defines a map from a two-sphere (S^2) to a two sphere. All these kinds of maps form the second homotopy group of the sphere, $\Pi_2(S^2)$, which is isomorphic to the group of integers, Z, under addition. The stability of the monopole can also be understood by the topological classification. The winding number does not change under the deformations of the Higgs field which are in the same homotopy of the Higgs field. These deformations could be either one of these i)time development of the Higgs field ii) continuous gauge transformations and iii) changing the surface (S) within the Higgs vacuum. As a consequence we get the quantization condition $ge = -4\pi n$. Observe that this differs from the Dirac quantization condition by a factor of two which can be attributed to the fact that we are able to put fractionally charged particles in the fundamental representation.

As mentioned earlier we want to calculate the mass of a monopole or in general a dyon. This can be done by looking at the (0,0) component of the energy momentum tensor. In doing so one gets

$$M = \int d^3x \left(\frac{1}{2}[(E_i^a)^2 + (B_i^a)^2 + (D_i\Phi^a)^2 + (D_0\Phi^a)^2] + V(\Phi) \right) \tag{10}$$

Using the equations of motion one can write the electric and magnetic charges as

$$g = \frac{1}{v}\int d^3x B_i^a (D^i\Phi)^a \qquad\qquad q = \frac{1}{v}\int d^3x E_i^a (D^i\Phi)^a \tag{11}$$

Introducing $\tan\theta = q\backslash g$ we can write the mass formula in the following form.

$$\begin{aligned} M &= \int d^x \left(\frac{1}{2}\{(E_i^a - D_i\Phi^a\sin\theta)^2 + (B_i^a - D_i\Phi^a\cos\theta)^2 + (D_0\Phi^a)^2\} + V(\Phi) \right) \\ &\quad + v(q\sin\theta + g\cos\theta) \end{aligned} \tag{12}$$

The first line is a positive semi-definite quantity so we get the Bogomol'nyi bound for the mass of a dyon

$$M \geq v\sqrt{g^2 + q^2} \tag{13}$$

This bound is saturated in the so called Bogomol'nyi-Prasad-Sommerfield **(BPS)** limit where the Higgs field is not allowed to self-interact i.e ($\lambda = 0$) and the scale invariance is broken by imposing the constraint $\Phi_{vac}^2 = v^2$. It is clear that the first line has to be identically zero. The states which saturate this bound are called BPS states. One remarkable fact is that this bound is universal in the sense that both elementary particles and the solitons in this theory obey the same mass formula. Based on this observation and Dirac's electric magnetic duality **Montonen** and **Olive** made a fascinating conjecture that there are two equivalent formulations of the Georgi-Glashow model. One is the electric formulation where Gauge bosons are fundamental particles and the monopoles are solitons. In the magnetic formulation monopoles are elementary particles and the familiar gauge bosons are extended objects. The lagrangians are essentially in the equal forms but one has to make the following substitution. $q \rightarrow g = \pm\frac{4\pi}{q}$. As it is clear this suggests that weak coupling regime in one theory corresponds to the strong coupling regime in the other formulation of the theory. One needs to be very careful here. This duality (symmetry) is not a symmetry of the model at hand instead it gives a new formulation of the theory. And it is not clear at all how it could be any useful in getting a better understanding of this particular model. This conjecture passes a couple of immediate tests. For example in the magnetic formulation one observes that two monopoles should not exert a force on each other and in the electric formulation this corresponds to non-interaction of two same charge vector bosons. This of course is true because repulsive Coulomb interaction of two vector particles is exactly canceled by the attractive Higgs exchange. (Remember that Higgs particle is massless in the BPS). There are a three immediate questions one can raise about this conjecture.

1) What will happen in the quantum theory? (Even if we set the quartic coupling to be zero an effective potential is generated by quantum corrections as in the case of Coleman-Weinberg mechanism. And this will ruin the universal mass formula.)

2) For duality to occur the monopoles should be vector particles. How can we assign spin-1 to monopoles?

3) We have totally ignored the dyons. Why not construct a theory where dyons are fundamental?

Introducing supersymmetry will cure the first two problems. In the extended ($N > 1$) SUSY theories BPS mass bound naturally arises as the central extension of the SUSY algebra. The answer for the third question will lead us to the idea of S-duality where the gauge group is enlarged to a subset of $SL(2, Z)$.

Strictly speaking the only theory that can have the exact electric-magnetic duality is $N = 4$. All the particles in this theory are in the adjoint representation and the theory is exactly finite. But this theory is almost unique and not rich enough to accommodate the phenomenology. $N = 2$ SUSY which is special-finite was shown by Seiberg and Witten to have an effective electric magnetic duality.

SUPERSYMMETRY

Haag-Lopuszanski and Sohnius found the most (up to isomorphisms) extended SUSY algebra by adding central charges. For example in N=2 the supercharges obey the following algebra

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J \\ \{Q^I_\alpha, Q_{\beta J}\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{IJ} Z \\ \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}J}\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{IJ}\bar{Z} \end{aligned} \tag{14}$$

I, J run from 1 to 2 , early Greek indices are spinorial and μ is the space time index. There is only one complex central charge for $N = 2$ and it is denoted by Z. By redefining the supercharges one can obtain the following algebra.

$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2}Z) \qquad \{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta}(M - \sqrt{2}\bar{Z}) \tag{15}$$

Where a_α and b_β are defined by

$$a_\alpha = \frac{1}{2}\{Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\}, \qquad b_\alpha = \frac{1}{2}\{Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\} \tag{16}$$

The other anti-commutation relations vanish. All physical states have positive definite norm so for massless states central charge is zero and only half of the supercharges ($a_\alpha s$) survive. For massive states there is a bound on their mass i.e $M \geq \sqrt{2}|Z|$. And it is clear that if the bound is saturated half of the supercharges are trivially realized and we have a short multiplet. One can construct the representations of this short multiplet as in the case of a fermionic osscilator. Assuming that there exits a **Clifford Vacuum** we have four states

$$|0> \qquad a^+_\alpha|0> \qquad a^+_\alpha a^+_\beta|0> \tag{17}$$

It is a remarkable fact that extended SUSY with non-vanishing central charge can have equal number of states as in the case of massless representation. This fact allows a consistent **Supersymmetric Higgs mechanism**.

Olive and Witten showed that one can identify the central charge in this theory by electric and magnetic charges. The precise relation is

$$Z = v(n_e + \tau n_m), \quad \text{where,} \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{18}$$

In this equation n_e, n_m are integers and denote electric and magnetic charges respectively. Some explanation is in order here. The complexified coupling arises because 'as it was shown by Witten , addition of a topological term , $\theta F\tilde{F}$ which is proportional to instanton number, gives electric charge to a monopole. The physics is periodic in θ with a period of 2π . So we have a duality transformation

$$T: \qquad \tau \to \tau + 1 \tag{19}$$

The Montonen-Olive duality transformation takes the following form

$$S: \qquad \tau \to -\frac{1}{\tau} \tag{20}$$

These two transformations generate the group $SL(2, Z)$ which can be denoted as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad a, b, c, d \in Z \text{and} \quad ad - bc = 1 \tag{21}$$

Since $g^2 > 0$ τ lives in the upper complex plane, $Im\tau \geq 0$ which is consistent with the positivity of the kinetic energy terms in the Lagrangian.

So we see that the BPS states quite naturally arise in $N = 2$ SUSY. And this is not a semi-classical result. If the bound is saturated at the tree level it has to remain so at the quantum level because quantum corrections are not supposed to change the number of degrees of freedom.

Here I would like to go one step back and introduce $N = 1$ supersymmetry and $N = 1$ symmetric non-linear sigma model. In addition to four dimensional space time one introduces two Grassman numbers , θ^α and $\theta_{\dot{\alpha}}$ and obtains the so called $N = 1$

superspace. A general superfield is a complex scalar function $\Phi(x^\mu, \theta^\alpha, \theta_{\dot\alpha})$. This function has a finite Taylor expansion in powers of the Grassmanian numbers. One can look at the effect of a rigid $N = 1$ supersymmetry transformation on this superfield after Taylor expansion. It turns out that a general superfield is not an irreducible representation of $N = 1$ supersymmetry. Two of the irreducible representations which can be obtained by putting convenient constraints are a chiral superfield and a vector superfield. Without going any further I will write down an $N = 1$ supersymmetric gauged sigma model Lagrangian which will be helpful to us when we construct the $N = 2$ theory.

$$\begin{aligned} \mathcal{L} &= \int d^4\theta K(\Phi^i, {\Phi^j}^\dagger e^{2V}) - \{\int d^2\theta W(\Phi^i) + h.c\} \\ & \frac{1}{8\pi} Im[\tau \int d^4x \int d^2\theta tr f(\Phi^i) W^\alpha W_\alpha] \end{aligned} \tag{22}$$

Where Φ is a chiral superfield (not to be confused with a general one) and V is a vector superfield and W_α is a spinor superfield. $K(\Phi^i, {\Phi^j}^\dagger e^{2V})$ is called the Kahler potential and $W(\Phi)$ is the superpotential. $f(\Phi)$ is called the gauge kinetic function. The theory defined by the above Lagrangian is a non-renormalizable one. This Lagrangian can be used as an effective Lagrangian. The particle contend of $N = 2$ theory we are about to study is same as $N = 1$ theory. There are two main differences in the lagrangians first $N = 2$ theory does not allow a superpotential and secondly there is a holomorphic function $\mathcal{F}$,called the pre-potential, in $N = 2$ theory which relates the Kahler potential and gauge kinetic function.

By using superspace formulation we can construct a Lagrangian that is symmetric under $N = 2$ SUSY algebra. $N = 2$ superspace is a 12 dimensional manifold whose coordinates are $(x^\mu, \theta, \bar\theta, \tilde\theta, \bar{\tilde\theta})$. We want to consider $N = 2$ SUSY without matter, so the particle contend of the theory is

$$\text{spin } 1 \quad A^i_\mu \qquad \text{spin } \frac{1}{2} \quad {\lambda_\alpha}^i \text{ and } \psi^{\beta i} \qquad \text{spin } 0 \quad \phi^i \tag{23}$$

All the fields are in the adjoint representation of the gauge group $SU(2)$ Classically the theory has a full global symmetry of $SU(2)_R \times U(1)_R$. But the instantons break the $U(1)_R$

to a discrete subgroup Z_8. And the true symmetry of the theory becomes

$$\frac{SU(2)_R \times Z_8}{Z_2} \tag{24}$$

(Here we took modulo Z_2 because center of $SU(2)$ includes Z_2.) Further more due to the Higgs mechanism Z_8 is spontaneously broken down to Z_4. Without going into detail I would like to write down an N=2 supersymmetric non-linear sigma model Lagrangian. It is given by

$$\mathcal{L} = \frac{1}{16\pi} Im \int d^4x d^2\theta d^2\tilde{\theta} \mathcal{F}(\Psi) \tag{25}$$

here $\mathcal{F}$ is called the $N = 2$ pre-potential and depends holomorphically on the $N = 2$ chiral superfield Ψ. This model in general is not renormalizable. But if $\mathcal{F}$ is quadratic in the chiral superfield Ψ; one obtains the super Yang-Mills theory.

MODULI SPACE OF $N = 2$ THEORY

Following Seiberg and Witten we want to calculate the low energy effective action starting from the microscopic theory. In a theory where massless particles are present one distinguishes between the so called **Wilsonian** effective action and the standard effective action which is obtained after Legendre transforming the generating functional of one-particle -irreducible diagrams. To obtain the Wilsonian effective action one introduces an IR cut off and the action holomorphically depends on this cut off. This is not the case for the standard effective action.

The microscopic $N = 2$ theory is given by

$$\begin{aligned}\mathcal{L} \quad = \quad & \frac{1}{g^2} Tr(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + g^2 \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}[\phi^\dagger, \phi]^2 \\ & -i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi - i\sqrt{2}[\lambda, \psi]\phi^\dagger - i\sqrt{2}[\bar{\lambda}, \bar{\psi}]\phi)\end{aligned} \tag{26}$$

The classical potential is

$$V = \frac{1}{2g^2} Tr([\phi^\dagger, \phi]^2) \tag{27}$$

The Higgs vacuum is defined by $V = 0$. Assuming that the gauge group is $SU(2)$ one sees that ϕ takes values in the 1 dimensional Cartan subalgebra which generate the group $U(1)$.

One can take $\phi = \frac{1}{2}a\sigma_3$. Here **a** is a complex number. Different values of a correspond to physically different theories and in this way one obtains a 1 dimensional complex manifold of gauge inequivalent vacua which is called the **moduli space** . Mathematically what we have done is to identify the vacuum with the coset space $\frac{SU(2)}{U(1)}$. But there are Weyl reflections , i.e $a \to -a$ which are elements of this coset space but do not take ϕ out of the Cartan subalgebra. So a better parameterization of the vacuum is given by the following parameter.

$$u =< tr\phi^2 > \qquad , \qquad < \phi >= \frac{1}{2}a\sigma_3 \tag{28}$$

We are going to see that the moduli space ,u-plane has some singularities which will lead us to the determination of the exact Wilsonian potential.

Calculation of the Witten index shows that $N = 2$ supersymmetry is not broken. For a non-vanishing $< \phi >$ due to the Higgs mechanism the vector mesons A^b_μ corresponding to $b = 1, 2$ become massive. And due to the unbroken supersymmetry ψ^b, λ^b ,for $b = 1, 2$ become massive as well. The remaining particles A^3_μ, ψ^3 and λ^3 and the and the scalar excitation in σ^3 direction are all massless and at low energies one can think of an $N = 2$ supersymmetric Abelian strongly interacting gauge theory. This theory need not be renormalizable of course. The form of the theory in $N = 1$ language is given by

$$\mathcal{L}_{eff} = \frac{1}{4\pi}Im(\int d^4\theta \frac{\partial \mathcal{F}}{\partial A}\bar{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A \partial A}W^\alpha W^\alpha) \tag{29}$$

where A is a chiral super field and "a" is the scalar component of it . The metric on the field space is given by

$$ds^2 = Im(\tau) da d\bar{a} \qquad \text{where} \qquad \tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2} \tag{30}$$

This is an asymptotically free theory and at high energies perturbative calculations are reliable. Exact form the perturbative pre-potential can be calculated to be

$$\mathcal{F}_{\text{perturbative}} = \frac{i}{2\pi}A^2 \ln \frac{A^2}{\Lambda^2} \tag{31}$$

There are different ways to derive this result. Following Seiberg one can show that this is the only holomorphic object that is consistent with $U(1)_R$ chiral symmetry which remains unbroken in the perturbative regime. Λ is a dynamically generated scale. This result comes only from the first loop calculations which is a property of $N = 2$ SUSY. Seiberg was the first to notice that pre-potential will receive non-perturbative corrections if one has to deal with the strong coupling regime. The form of the non-perturbative corrections is dictated by the behavior of the theory near the semi-classical region ($A \to \infty$) and the remaining Z_4 symmetry. So for a generic A we have

$$\mathcal{F} = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty} F_k (\frac{\Lambda}{A})^{4k} A^2 \tag{32}$$

F_1 was calculated to be non-zero. Seiberg and Witten determined all the F_k s .From the above form of the pre-potential one observes that $\tau(a)$ can not be defined globally. For example, at large a one obtains $\tau(a) \sim \frac{i}{\pi} \ln(\frac{a^2}{\Lambda^2}) + 3$. Since $\mathrm{Im}\tau(a)$ is an harmonic function it cannot have a global minimum so it cannot be positive everywhere if it is globally defined. This is a curious result which leads us to find different descriptions of the theory in different parts of the moduli space.

DUALITY

We define $a_D = \frac{\partial \mathcal{F}}{\partial a}$ so that the moduli space metric becomes

$$(ds)^2 = \mathrm{Im} da_D d\bar{a} = -\frac{i}{2}(da_D d\bar{a} - da d\bar{a_D}) \tag{33}$$

which is completely symmetric between a and a_D. So instead of a we can use a_D to parameterize the moduli space with a different harmonic function replacing $\mathrm{Im}(\tau)$

What we mean by a dual description of a theory can be understood from the following example. In the electric formulation and $N = 1$ language we have

$$\frac{1}{8\pi} Im \int d^2\theta \tau(A) W^2 \tag{34}$$

By implementing the Bianchi identity $ImDW = 0$ as a Lagrange multiplier and integrating it out one obtains the magnetic formulation

$$\frac{1}{8\pi} Im \int d^2\theta (-\frac{1}{\tau(A)} W_D^2) \tag{35}$$

which exactly states that electric gauge fields are replaced by magnetic gauge fields which couple to magnetic charges. So the $N = 2$ low energy effective Lagrangian can be written in the form

$$\frac{1}{4\pi} Im (\int d^4\theta \frac{\partial \mathcal{F}(A_D)}{\partial A_D} \bar{A_D} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial A_D \partial A_D} W_D^\alpha W_D^\alpha) \tag{36}$$

We have found earlier that the BPS-saturated states satisfy $M = \sqrt{2}Z$. By coupling our theory to a hyper-multiplet of two $N = 1$ chiral superfields $M, \tilde{M}$ with electric charge n_e we can find what Z is. The only $N = 2$ supersymmetric coupling is of the Yukawa form which is

$$\sqrt{2} n_e A M \tilde{M}$$

So one can identify (for purely electric states) $Z = a n_e$. By the duality arguments above for dyons one obtains

$$Z = a n_e + a_D n_m \tag{37}$$

MONODROMIES ON THE MODULI SPACE

1) Monodromy at large u

At large $|a|$ the theory is asymptotically free and $u = \frac{1}{2} a^2$. Instantons do not contribute and one can assume that pre-potential is given by the one-loop formula so

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \ln(\frac{a}{\Lambda} + \frac{ia}{\pi}) \tag{38}$$

A close loop at $u \to \infty$ shows that

$$a_D \to -a_D + 2a$$

$$a \to -a$$

so that the monodromy matrix is

$$M_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

under this transformation $(n_m, n_e) \to (-n_m, -n_e - 2n_m)$ so that the mass of the BPS states do not change.

2) Monodromies at finite u

The existence of a singularity at $u = \infty$ implies that there are extra singularities somewhere else. (Otherwise one can continuously deform the theory to the weakly coupling regime.) Seiberg-Witten conjectured that there are two extra singularities in the moduli space at finite u. And the singularity at $u = 0$ in the classical moduli does not exist in the quantum moduli. A singularity on the moduli space corresponds to some massless particle. The first guess is that some vector bosons become massless at these points. But as SW argued this does not lead to a consistent picture. The only other massive states in the theory are spin $\leq \frac{1}{2}$ monopoles and dyons. We assume that these particles becomes massless at $u = \pm 1$. And to find the monodromy matrices at these points we need to use the dual formulation of the theory. Considering a point u_0, on the moduli space, where a monopole becomes massless so that we have $a_D(u_0) = 0$. The low energy theory (after integrating out the massive monopoles around this point) is a weakly interacting Abelian theory. By using the one loop beta function one can obtain

$$\tau_D \sim -\frac{1}{\pi} \ln a_D \tag{39}$$

Since a_D is a good coordinate around the singularity one can make an ansatz as $a_D \sim (u - u_0)$. Once again by moving u around a contour we can find the monodromy matrix

$$M_1 = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

The third singularity can be found directly by assuming a dyon becomes massless at a point on the moduli space. One obtains

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

The assumption that there are only two extra singularities was later shown to be true.

MONOPOLE CONDENSATION AND CONFINEMENT

Confinement of color has been one of the greatest puzzles for the strongly interacting theories. For example QCD Lagrangian has quarks and gluons as dynamical fields but experiments show no evidence for free quarks and gluons. These fields are permenantly confined. We know , by theoretical arguments and by lattice calculations that there is infra red slavery for QCD , the perturbative coupling constant becomes unboundedly large for small energies. Lattice gauge theory calculations , by using Wilson line arguments, indicate that the area law occurs and the potential grows linearly which signals the confinement. But still a fundamental understanding of the dynamics of confinement in QCD is lacking. But as we will show below a clear understanding of confinement has been achieved in the Seiberg-Witten model. Although supersymmetry is a vital ingredient in this model one hopes that similar ideas could be applied to QCD.

Dual Meissner Effect

It is well known that in a strongly interacting theory confinement of electric charge will take place as a consequence of dual-Meissner effect. One expects the confinement to take place where condensation of monopoles occur. After softly breaking $N = 2$ to $N = 1$ SW showed that one can understand the confinement. First I would like to briefly discuss the standard Meissner effect and confinement of magnetic charge. A system of charged scalar particles coupled to electromagnetic fields is described by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu(\phi))^\dagger(D^\mu(\phi)) - \mu^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{40}$$

Here the covariant derivative is given by $D_\mu = \partial_\mu - i2eA_\mu$ (The factor 2 is put in anticipation for Cooper pairs.) If μ^2 is positive the $U(1)$ symmetry of the Lagrangian is broken and the gauge field becomes massive. In a static gauge configuration ($A_0 = 0$) one obtains the following equations for the magnetic field.

$$\nabla^2\vec{B} - m^2\vec{B} = 0, \qquad \vec{\nabla}\wedge\vec{B} + m^2\vec{A} = 0 \tag{41}$$

Here m is the mass of the Higgs field. The first equation states that magnetic field is expelled from the region (superconductor) except for a small penetration depth. This is the Meissner effect. The second equation is known as the London current and it implies the existence of a steady current even-tough there is no electric field.

What has happened here is quite fascinating. The vacuum of the theory consists of pairs of electrically charged particles (condensation of electric charges) and this leads to the expelling of magnetic fields. If we put two magnetic probes in this medium due to the conservation of flux not all the magnetic fields can be expelled and this will result a formation of flux tube between the magnetic probe charges. One can show that the potential grows linearly. It takes an infinite amount of energy to separate these to magnetic charges so they are confined.

Confinement of electric charge can be understood in a similar way. When a magnetically charged field condenses magnetic field field is completely screened and if we put two electrically charged particles in this magnetic vacuum there will be a formation of electric flux tube connecting the charges. The potential grows linearly and the confinement of charge is realized. This is called the Dual Meissner Effect.

$N = 2$ theory, as we have considered earlier, consists of an Abelian vector multiplet in the semi-classical region. We softly break $N = 2$ down to $N = 1$ by turning on a small mass term ,i.e $mTr\Phi^2$, for the scalar multiplet. This moves the vacuum degeneracy and gives mass to the scalar multiplet. Near the points where the monopoles are massless the $N = 1$ superpotential takes the form ;

$$\hat{W} = \sqrt{2} A_D M \tilde{M} + mU(A_D) \tag{42}$$

where $M, \tilde{M}$ describe the monopole hyper-multiplet. Assuming that m is nonzero the vacua corresponds to

$$M = \tilde{M} = (\frac{-mu'(0)}{\sqrt{2}})^{\frac{1}{2}} \tag{43}$$

Non-vanishing of M states that massless monopoles condense in the vacuum so the electric charges in the theory are confined.

SOLUTION OF THE MODEL

We have found the monodramies of the multi-valued functions , $a_D(u)$ and $a(u)$ around certain singularities. By using this information we can determine the functions uniquely up to multiplications by entire functions. This is called a Riemann-Hilbert problem. The entire functions can be determined by looking at the asymptotic limits.

The monodromy matrices ;M_∞ and M_1 generate $\Gamma(2)$, a subgroup of $SL(2,Z)$. This group is defined to be

$$\Gamma(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \quad b = 0 mod 2\}$$

$\Gamma(2)$ is duality symmetry in the moduli space which includes strong- weak coupling symmetry. So the the quantum moduli space can be defined as;

$$\mathcal{M}_q = H^+ \backslash \Gamma(2) \tag{44}$$

where H^+ denotes the upper half plane due to the positivity of the moduli τ. By an ingenious choice Seiberg-Witten introduced a genus-one Riemann surface (elliptic curve) which is parameterized by $\mathcal{M}_q$. The curve is

$$y^2 = (x-1)(x+1)(x-u) \tag{45}$$

Here u is a coordinate on $\mathcal{M}_q$ and for every u there exists a genus-one Riemann surface. This curve has the correct symmetries of our moduli space. By looking at the branch cuts and the non-singularity of the curve one can easily see that it is a torus.

How can we identify a_D and a on this torus?

Let (α, β) be the canonical basis for the first **homology group** (closed cycles which are not exact) of the torus. We can define a scalar product of these cycles with the elements of the first **cohomology group** in the following way;

$$(\alpha, \omega) = \oint_\alpha \omega \tag{46}$$

Cohomology group consists of (cocycles) differential forms which are (co)closed and are not (co)exact. As vector spaces cohomology and homology groups are isomorphic and so they have the same number of basis. One can choose the following two holomorphic differentials as a basis for the cohomology group

$$\omega_1 = \frac{dx}{y} \qquad\qquad \omega_2 = \frac{xdx}{y} \tag{47}$$

The torus can be characterized by

$$\tau_u = \oint_\alpha \omega_1 \backslash \oint_\beta \omega_1 \tag{48}$$

It is straight forward to show that $\tau = \tau_u$ So more explicitly what we have is

$$\tau = \frac{da_D \backslash du}{da \backslash du} = \oint_\alpha \omega_1 \backslash \oint_\beta \omega_1 \tag{49}$$

By using the asymptotic behavior of the a_D, a we can determine ω_1 exactly.

$$\omega_1 = \frac{\sqrt{2}}{2\pi} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} \tag{50}$$

To determine a and a_D we have to specify the cycles α and β. We choose α to be a loop around the points 1 and u. And β is chosen to be a loop around 1 and -1. Thus we obtain

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} \qquad\qquad a_D(u) = \frac{\sqrt{2}}{\pi} \int_{1}^{u} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} \tag{51}$$

These integrals can be solved in terms of hyper-geometric functions. It can be easily checked that these functions have the correct monodramies. So we have obtained the metric on the moduli space.

Seiberg-Witten solution was immediately generalized to larger gauge groups and theories which have matter hyper-multiples. Some explicit multi-instanton calculations show that the solution is correct.

The impact of the Seiberg-Witten theory on the classification of four manifolds was huge. Donaldson, early in the 80s, used self-dual Yang-Mills equations to find some topological invariants (essentially correlation functions) of four manifolds. Since the the group in the moduli space is non-Abelian it is a very difficult task to do any calculation. Witten used the duality idea we discussed above to find the topological invariants. It is clear that since these invariant are metric-independent one can perform the calculation either in the infrared or ultraviolet. Well we have shown that an strongly interacting non-Abelian theory of electric charges can be described as a weakly interacting theory of monopoles. So one can perform the calculations in the weak theory easily.

DYON CONDENSATION PROBLEM

In the above analysis we did not look carefully at the moduli space of the vacua where dyons become massless. As a research problem I will be examining in general the problem of dyon condensation in field theories. As a starting problem , though it may turn out to be unrelated , I will analyze the following problem.

In a very interesting paper Hosotani showed that 3 dimensional Georgi-Glashow model is dual to the Josephson effect. Or maybe it is better to put it in this form Josephson effect has an underlying field theory which is 3 dimensional compact QED. But one delicate point is that electric fields in one theory corresponds to magnetic fields in the other theory. It is not clear to me at this point why such an intricate duality exists. I would like to understand in Josephson effect what would correspond to the addition of a Chern-Simons term in the Georgi-Glashow model.

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References

[1] N. Seiberg, and E. Witten *Nucl. Phys.* B**426** (1994) 19 ;*Nucl. Phys.* B**431** (1994) 484

[2] D. Olive, Exact Electromagnetic Duality, `hep-th/9508089`

[3] A. Bilal, Duality in $N = 2$ SUSY $SU(2)$ Yang Mills Theory : A pedagogical introduction to the work of Seiberg and Witten , `hep-th/9601007`

[4] P. Di Vecchia, Duality in supersymmetric gauge theories, `hep-th/9608090`

[5] L. A Gaume, and S. F. Hassan, Introduction to S-Duality in N=2 Supersymmetric Gauge Theories, `hep-th/9701069`

[6] J. A Harvey, Magnetic monopoles, duality and supersymmetry `hep-th/9603086`

[7] S. V Ketov, Solitons, monopoles and duality: from sine-Gordon to Seiberg- Witten, `hep-th/9611209`

[8] J. D Lykken, Introduction to supersymmetry `hep-th/9612114`

[9] W. Lerche, Introduction to Seiberg-Witten theory and its stringy origin, `hep-th/9611190`

[10] M. Shifman, Non-perturbative dynamics in supersymmetric gauge theories, `hep-th/9704114`

[11] M. Peskin, Duality in supersymmetric Yang-Mills theory, `hep-th/9702094`

[12] E. Witten, Monopoles and Four Manifolds, `hep-th/9411102`

[13] Y. Hosotani *Phys. Lett.* B**69** (1977) 499