ADAPTIVE BACKSTEPPING CANCELLATION OF UNMATCHED UNKNOWN SINUSOIDAL DISTURBANCES FOR LTI SYSTEMS BY STATE DERIVATIVE FEEDBACK

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ABSTRACT
Solutions already exist for the problem of canceling sinusoidal disturbances by measurement of the state or by measurement of an output for linear and nonlinear systems. In this paper, we design an adaptive backstepping controller to cancel unmatched sinusoidal disturbances forcing a linear time-invariant system which is augmented by a linear input subsystem by using only measurement of state-derivatives of the original subsystem and state of the input subsystem. Our design is based on four steps, 1) parametrization of the sinusoidal disturbance as the output of a known feedback system with an unknown output vector that depends on unknown disturbance parameters, 2) design of an adaptive disturbance observer for both disturbance and its derivative, 3) design of an adaptive controller for virtual control input, and 4) design final controller by defining error system and using backstepping procedure. We prove that the equilibrium of the closed-loop adaptive system is stable and state of the considered error system goes to zero as $t \to \infty$ with perfect disturbance estimation. The effectiveness of the controller is illustrated with a simulation example of a third order system.

1 Introduction
The problem of canceling sinusoidal disturbances in dynamical systems is a fundamental control problem, with many applications such as vibrating structures [1], active noise control [2] and rotating mechanisms control [3]. The common method to approach this problem is the internal model principle for which a general solution is given in [23], [24] in the case of linear systems. In the internal model approach, the disturbance is modeled as the output of a linear dynamic system which is called an exosystem. Then the effect of the disturbance on the plant response can be completely compensated by adding a replica of the exosystem model in the feedback loop.

The output regulation problem for minimum phase, uncertain nonlinear systems is solved in [4], [7] and extended for nonminimum phase plants in [6]. The regulation of a linear time-varying system is considered in [12], and the regulation problem for time-varying known exosystem is studied in [8]. On the other hand, disturbance cancelation designs also exist for continuous-time linear systems [5], [11], [13], [21] and discrete-time linear systems [16]. Moreover, designs for nonlinear systems are proposed in [9], [10], [19], [22]. In all of these references, the controllers are designed by using measurement of state or an output.

In the last decade, the state derivative feedback control has drawn the attention of many researchers [26]–[31] due to its various advantages in applications. In most practical problems, especially disturbance cancelation problems, using accelerometers as sensors is easier, cheaper and more reliable than using position sensors. In this case, from the signals of the accelerometers it is possible to obtain the velocities with a sufficient precision but not the displacements. Therefore, the system can be modeled by considering position and velocity as states and the state-derivatives are available for control design. A control design by state-derivative feedback for known linear time-invariant systems with matched unknown sinusoidal disturbances for known and unknown system parameters are proposed in [32], [33].

Heave control of surface effect ship motivates us to consider this problem structure. In [25], dissipative control is proposed for surface effect ships. In this particular example, states of the sys-
tem are heave, heave rate and pressure. We replace the position sensor with an accelerometer and consider pressure as a virtual controller since we are not able to actuate pressure directly. The real actuator of the all system is considered as mass flow rate.

We extend the result in [32] by relaxing the matched disturbance condition. Employing an approach inspired by [17], we parameterize the unknown sinusoidal disturbance and design an adaptive backstepping controller by using state derivative of main system and state of the input subsystem to cancel unmatched unknown sinusoidal disturbances forcing linear time-invariant systems. We prove that the equilibrium of the closed loop system is stable and the states of the considered error system go to zero as $t \rightarrow \infty$ with perfect disturbance estimation.

In Section 2, we introduce the problem and state our main stability theorem. In Section 3, we prove the theorem. A simulation example is presented in Section 4.

2 Problem Statement and Adaptive Controller Design

We consider the single-input LTI system

$$
\dot{x} = Ax + B(p + v) \\
\dot{p} = a^T x + b_p p + b_u u
$$

(1)

with the state $x \in \mathbb{R}^n$ and $p \in \mathbb{R}$, input $u \in \mathbb{R}$, and sinusoidal disturbance $v \in \mathbb{R}$ given by

$$
v(t) = \sum_{i=1}^{q} g_i \sin(\omega_i t + \phi_i),
$$

(2)

where $i \neq j \Rightarrow \omega_i \neq \omega_j$, $\omega_i \in \mathbb{Q}$, $g_i, \phi_i \in \mathbb{R}$.

The sinusoidal disturbance $v$ can be represented as the output of a linear exosystem,

$$
\dot{w} = Sw \\
v = h^T w
$$

(3)-(4)

where $w \in \mathbb{R}^{2q}$ and the choice of $S \in \mathbb{R}^{2q \times 2q}$ and $h \in \mathbb{R}^{2q}$ is not unique.

We make the following assumptions regarding the plant (1) and the exosystem (3)-(4):

Assumption 1. $A$ is invertible.

Assumption 2. The pair $(A, B)$ is controllable.

Assumption 3. $b_u \neq 0$.

Assumption 4. $x$ and $v$ are not measured but $\dot{x}$ and $p$ are measured.

Assumption 5. The pair $(h^T, S)$ is observable.

Assumption 6. The eigenvalues of $S$ are imaginary, distinct and rational.

Assumption 7. $q$ is known.

Assumption 8. $S$ and $h$ are unknown.

Assumption 9. $g_i \neq 0$ for all $i \in \{1, \ldots, q\}$

Under Assumptions 1 and 2, there exists a control gain $K \in \mathbb{R}^{1 \times n}$ such that $(A^{-1} + A^{-1} BK)$ is Hurwitz [26].

We state now our adaptive controller with a disturbance observer. In Section 3 we analyze the stability properties of the closed-loop system.

The adaptive controller for the system (1), (3), (4) is given by

$$
u = \frac{1}{1 + KB} b_u \left( (\hat{\theta}^T I - (1 + KB)(b_p - a^T A^{-1} B)) p - (\hat{\theta}^T N + (\hat{\theta}^T N + K)A + (1 + KB)a^T A^{-1} B) \hat{x} + (1 + KB)(a^T A^{-1} B) \hat{\theta}^T \xi - (KB + \hat{\theta}^T I) \hat{\theta}^T \xi - \hat{\theta}^T \eta - \hat{\theta}^T \hat{\eta} + - ((KB + \hat{\theta}^T I)^2 + c) e \right),
$$

(5)

where $c > \frac{1}{2}$ and

$$
e = p - (-K \hat{x} - \hat{\theta}^T \hat{\xi}).
$$

(6)

The update laws for $\hat{\theta}(t)$ and $\hat{\theta}(t)$ are given by

$$
\dot{\hat{\theta}} = -\gamma_a (A^{-1} B)^T P \hat{x} + (1 + KB)(a^T A^{-1} B) e,
$$

(7)

$$
\dot{\hat{\theta}} = \gamma_b (KB + \hat{\theta}^T I)
$$

(8)

with $\gamma_a, \gamma_b > 0$ and the positive definite matrix $P$ is a solution of the matrix equation

$$
(A^{-1} + A^{-1} BK)^T P + P(A^{-1} + A^{-1} BK) = -2I.
$$

(9)

The disturbance observer is given by

$$
\eta = G(\eta + N(\dot{x} - Bp)) - NA \hat{x}
$$

(10)

$$
\xi = \eta + N(\dot{x} - Bp),
$$

(11)

where $G$ is a $2q \times 2q$ Hurwitz matrix with distinct poles and constitutes a controllable pair with a chosen vector $l \in \mathbb{R}^{2q}$ and $N$ is a $2q \times n$ matrix which is given by

$$
N = \frac{1}{B^T B} \hat{b}_u^T,
$$

(12)
where the given \( N \) is one of the many solutions of the following equation

\[ NB = l. \quad (13) \]

Since the matrices \( G \) and \( S \) have disjoint spectra, the pair \((h^T, S)\) is observable, and the pair \((G, l)\) is controllable, the Sylvester equation

\[ MS - GM = lh^T, \quad (14) \]

has a unique solution [34]. This fact is exploited in the proof of our stability result (Lemma 1).

We first define the signals needed in the analysis and state a theorem describing our main stability result. Then we prove the theorem using a series of technical lemmas in Section 3.

Estimation errors of the unknown parameters are denoted by

\[ \hat{\theta}(t) = (MS)^{-T}h - \hat{\theta}(t), \quad (15) \]

\[ \hat{\beta}(t) = \frac{1}{1 - h^T(MS)^{-1}l}G^T(MS)^{-T}h - \hat{\beta}(t), \quad (16) \]

and \( \delta(t) \) and \( \xi(t) \) denotes the signals,

\[ \delta(t) = MS\nu(t) - \xi(t), \quad (17) \]

\[ \tilde{\xi}(t) = \bar{\xi}(t) - \xi(t), \quad (18) \]

where

\[ \bar{\xi}(t) = \int_0^t e^{G(t - \tau)} \nu(\tau) d\tau. \quad (19) \]

**Theorem 1.** Consider the closed-loop system consisting of the plant (1) forced by the unknown sinusoidal disturbance (4), the disturbance observer (10), (11) and the adaptive controller (5), (7), (8). Under Assumptions 1–8, the following holds:

(a) The equilibrium \( x = 0, \delta = \bar{\xi} = \hat{\theta} = \hat{\beta} = 0, e = 0 \) is globally stable.

(b) For all initial conditions \( x(0) \in \mathbb{R}^n, \bar{\xi}(0) \in \mathbb{R}^{2q}, \hat{\theta}(0) \in \mathbb{R}^{2q}, \hat{\beta}(0) \in \mathbb{R}^{2q}, e(0) \in \mathbb{R} \) and all \( w(0) \in \mathbb{R}^{2q} \) such that Assumption 9 holds, the signals \( x(t), e(t), \hat{\theta}(t), \delta(t), \tilde{\xi}(t), \nu(t) - \hat{\beta}^T(t)\tilde{\xi}(t) \) converge to zero as \( t \to \infty \).

**3 Stability Proof**

The following lemma enables us to represent the unknown sinusoidal disturbance as the output of a linear system whose input is the disturbance itself, whose state and input matrices are known, and whose output matrix is unknown.

**Lemma 1.** Let \( G \in \mathbb{R}^{2q \times 2q} \) be a Hurwitz matrix with distinct eigenvalues and let \((G, l)\) be a controllable pair. Then, \( \nu \) can be represented as the output of the model

\[ \dot{z} = Gz + l\nu \]

\[ \nu = \theta^T \dot{z} \]

\[ \theta^T = h^T(MS)^{-1}. \]

**Proof.** This result and its proof are inspired by [17]. To establish (20) from (3), consider

\[ z = Mw. \]

Differentiating (23), we obtain

\[ \dot{z} = MSw. \]

Using (14), we have

\[ \dot{\nu} = MS\nu - \dot{\nu} = MS\nu. \]

Substituting (4) and (23) into (25) yields (20). Substituting \( w = (MS)^{-1} \dot{\nu} \) into (4), we obtain (21) and (22).

The previous lemma enables us to write the unknown external disturbance \( \nu \) as the product of an unknown constant \( \theta \) and the vector \( \dot{z} \). However, \( \dot{z} \) is not accessible, since the signal \( \nu \) can not be measured. To overcome this problem, we design the observer (10)–(11).

The following lemma establishes the properties of the observer.

**Lemma 2.** The inaccessible disturbance \( \nu \) and \( \dot{\nu} \) can be represented in the form

\[ \nu = \theta^T \xi + \theta^T \delta, \]

\[ \dot{\nu} = \theta^T \xi + \beta^T \delta \]

where

\[ \beta = \frac{1}{1 - \theta^T G^T \theta} \]

\( \dot{\delta} \) in \( \mathbb{R}^d \) obeys the equation

\[ \dot{\delta} = G\delta. \]
Proof. Differentiating (20) with respect to time, we obtain
\[ \ddot{z} = G \dot{z} + N \dot{\nu}. \]  
(30)
Substituting (25) into (17) and using (4) and (20), we obtain
\[ \delta = \dot{z} - \dot{\xi}. \]  
(31)
Differentiating \( \delta \) with respect to time and in view of (30), (10) and (11), we get
\[ \dot{\delta} = G \ddot{z} + l \ddot{\nu} \]  
\[ + N A \dot{x} - N A \dot{x} - N B \nu. \]  
(32)
Substituting (11) into (32), using (31) and the fact that \( NB = I \), we get (29). Differentiating (21) and using (30), we obtain
\[ \dot{\nu} = \frac{1}{\lambda} \dot{\theta}_{l}^{T} G \ddot{z}. \]  
(33)
Using (21), (28), (31) and (33), we obtain (26) and (27). \[ \blacksquare \]

Lemma 3. There exists \( \rho > 0 \) such that for all \( t_{0} \geq 0 \), the following holds
\[ Q_{p}(\rho, t_{0}) = \int_{t_{0}}^{t_{0}+\rho} \dot{\xi}(t) \dot{\xi}^{T}(t) dt \]  
\[ - \frac{1}{\rho} \int_{t_{0}}^{t_{0}+\rho} \dot{\xi}(t) dt \int_{t_{0}}^{t_{0}+\rho} \dot{\xi}^{T}(t) dt > 0. \]  
(34)
Proof. By differentiating (31) with respect to time and using (29), (30), we obtain
\[ \ddot{\xi} = G \dot{\xi} + N \dot{\nu}, \]  
(35)
where
\[ \nu = \sum_{i=1}^{q} \overline{g}_{i} \cos(\omega_{i} t + \phi_{i}), \]  
(36)
with \( \overline{g}_{i} = g_{i} \omega_{i} \). By solving (35), we get
\[ \xi(\tau) = e^{G \tau} \xi(0) + \int_{0}^{\tau} e^{G(\tau-\sigma)} \dot{\nu}(\sigma) d\sigma. \]  
(37)
Since \( G \) has distinct eigenvalues and is Hurwitz, it is diagonalizable. Using a Jordan decomposition of the matrix \( G \), we can write
\[ G = L \Lambda L^{-1}, \]  
(38)
where \( L \) is the square \( 2q \times 2q \) matrix whose \( i \)th column is the \( i \)th eigenvector of \( G \) and \( \Lambda \) is the diagonal matrix whose diagonal elements are the corresponding eigenvalues of \( G \).

Defining \( L^{-1} l = \overline{L} \), substituting (38) into (37) and using the property \( e^{t \Lambda L^{-1}} = L e^{t \Lambda} L^{-1} \), we get
\[ \xi(\tau) = \overline{L} e^{t \Lambda} L^{-1} \xi(0) + L e^{t \Lambda} \int_{0}^{\tau} e^{-t \Lambda} \dot{\nu} d\sigma. \]  
(39)
By computing the integral in (39), we obtain
\[ \xi(\tau) = L \left( e^{t \Lambda} C_{e} + \Psi(\tau) \right), \]  
(40)
where \( \Psi \in R^{2q} \) is the vector whose \( j \)th row is
\[ \sum_{i=1}^{q} \frac{\overline{g}_{i}}{\lambda_{j} + \omega_{i}^{2}} \left( -\lambda_{j} \cos(\omega_{j} \tau + \phi_{j}) + \omega_{j} \sin(\omega_{j} \tau + \phi_{j}) \right), \]  
(41)
and
\[ C_{e} = L^{-1} \xi(0) - \Psi(0). \]  
(42)
Since \( \dot{\nu} \) is a sufficiently rich signal order of \( 2q \) and \( (G, l) \) is a controllable pair, \( \dot{\xi} \) is persistently exciting [37]. Therefore, there exist positive \( \rho^{*} \) and \( \alpha_{0} \) such that for all \( \rho > \rho^{*} \) and \( t_{0} \geq 0 \) the following holds
\[ \int_{t_{0}}^{t_{0}+\rho} \dot{\xi}(t) \dot{\xi}^{T}(t) dt \geq \rho \alpha_{0} I. \]  
(43)
Under Assumption 6, the frequencies of \( \dot{\nu} \) can be represented as
\[ \omega_{i} = \frac{\text{num}(\omega_{i})}{\text{den}(\omega_{i})}, \quad \text{num}(\omega_{i}), \text{den}(\omega_{i}) \in Z^{+}, \quad i = 1 \ldots q. \]  
(39)
Then \( \rho \) that is given by
\[ \rho = 0 \text{lcm}(\text{num}(\omega_{1}), \ldots, \text{num}(\omega_{q})) \]  
\[ \times \text{lcm}(\text{den}(\omega_{1}), \ldots, \text{den}(\omega_{q})) 2\pi > \rho^{*}, \]  
(44)
where \(\text{lcm}\) is the abbreviation of the least common multiple, satisfies (43) if \(\theta \in \mathbb{Z}^+\) is chosen sufficiently large for given \(\rho^\ast\) and \(\omega_1, \ldots, \omega_t\). Since \(\Psi(t)\) defined by (41) has a period \(\rho\) and incorporates only zero-mean functions, it follows that

\[
\int_{t_0}^{t_0+\rho} \Psi(t) \, dt = 0. \tag{45}
\]

Substituting (40)–(45) into (34), we get

\[
Q_p(t_0, t_0) \geq \Pi \Pi^T, \tag{46}
\]

where

\[
\Pi = \rho \bar{u}_0 - \frac{1}{\rho} \Gamma(\rho) \Gamma^T(\rho), \tag{47}
\]

\[
\bar{u}_0 = \alpha_0 / \lambda_{\max}\{LL^T\},
\]

\[
\Gamma(\rho) = \begin{bmatrix}
    c_1 e^{\lambda_1 \rho} (e^{\lambda_1 \rho} - 1) \\
    \vdots \\
    c_d e^{\lambda_d \rho} (e^{\lambda_d \rho} - 1)
\end{bmatrix},
\]

and \(c_i\) denotes the \(i\)th row of the vector \(C_i / \lambda_i\).

Since \(L\) is full rank, \(Q_p\) satisfies the inequality (34) if \(\mu^T \Pi \mu > 0\) for all nonzero \(\mu \in \mathbb{R}^{2d}\). Using (47), we have

\[
\mu^T \Pi \mu \geq \rho \bar{u}_0 (\mu_1^2 + \ldots + \mu_d^2) - \frac{1}{\rho} \left( \mu_1 c_1 e^{\lambda_1 \rho} (e^{\lambda_1 \rho} - 1) \right) + \ldots + \mu_d c_d e^{\lambda_d \rho} (e^{\lambda_d \rho} - 1) \right)^2. \tag{49}
\]

By using the Cauchy-Schwarz inequality and by noting that \(\lambda_i < 0, (e^{\lambda_i \rho} - 1) \leq 1\), we have

\[
\mu^T \Pi \mu \geq \rho \bar{u}_0 (\mu_1^2 + \ldots + \mu_d^2) - \frac{1}{\rho} \left( \mu_1 c_1 e^{\lambda_1 \rho} (e^{\lambda_1 \rho} - 1) \right)^2 + \ldots + \left( \mu_i c_i e^{\lambda_i \rho} (e^{\lambda_i \rho} - 1) \right)^2 \geq \left( \mu_1^2 + \ldots + \mu_d^2 \right) \left( \rho \bar{u}_0 - \frac{1}{\rho} \left( c_1 e^{\lambda_1 \rho} \right)^2 + \ldots + \left( c_d e^{\lambda_d \rho} \right)^2 \right). \tag{50}
\]

Since \(\rho \bar{u}_0\) is monotonically increasing and \(\frac{1}{\rho} \left( c_1 e^{\lambda_1 \rho} \right)^2 + \ldots + \left( c_d e^{\lambda_d \rho} \right)^2\) is monotonically decreasing with respect to \(\rho\) for all fixed \(t_0\), one can find a \(\rho\) using (44) such that for all \(t_0 \geq 0\), (34) holds.

**Proof of Theorem 1**: With the error variable (6), the closed-loop is written as

\[
\dot{x} = A_{cl} x + B \left( \theta^T \xi + \theta^T \delta + e \right), \tag{51}
\]

\[
\dot{e} = \left( (KB + \hat{b}^T l)^2 + c + \bar{B}^T P \bar{B} \right) e + (KB + \hat{b}^T l) \times \left( \beta^T \xi + \beta^T \delta \right) + \bar{B}^T P A_{cl} x, \tag{52}
\]

where

\[
A_{cl} = (A^{-1} + A^{-1} BK)^{-1}, \tag{53}
\]

\[
\bar{B} = A_{cl} \bar{B}. \tag{54}
\]

The stability of the equilibrium of the closed-loop system is established with the use of Lyapunov function

\[
V = \frac{1}{2} \dot{x}^T P x + \frac{1}{2} e^T + \frac{1}{2} \hat{\theta}^T \hat{\theta} + \frac{1}{2} \hat{\beta}^T \hat{\beta} + \frac{\varepsilon_\theta \varepsilon_\beta^T \varepsilon_\theta \varepsilon_\beta}{2} P \delta \tag{56}
\]

where

\[
G^T P_G + P_G G = -2 I, \tag{57}
\]

\[
\varepsilon_\delta = \lambda_{\max}(\theta \hat{\theta}^T P \hat{P} \hat{P} \theta^T) + ((1 + KB) a^T \bar{B})^2 \times \lambda_{\max}\{\theta \theta^T\} + \lambda_{\max}\{\beta \beta^T\}. \tag{58}
\]

Taking time derivative of \(V\), in view of (7), (8), (29), (51) and (52), we obtain

\[
\dot{V} = \dot{x}^T \dot{x} + \left( (KB + \hat{b}^T l)^2 + c \right) e^2 - (1 + KB) a^T \bar{B} \times \theta^T \delta e \theta^T \beta \theta^T \hat{P} \dot{x} + (KB + \hat{b}^T l) \beta^T \delta e - \varepsilon_\delta \delta^T \delta. \tag{59}
\]

Using Young’s inequality for the cross terms, we get

\[
\dot{V} \leq \dot{x}^T \dot{x} + \left( (KB + \hat{b}^T l)^2 + c \right) e^2 + \frac{1}{2} e^2 + ((1 + KB) a^T \bar{B})^2 \delta^T \theta \theta^T \delta + \frac{1}{2} \hat{x}^T \hat{x} + \frac{1}{2} \delta^T \beta \theta^T \beta \theta^T \delta + \frac{1}{2} (KB + \hat{b}^T l)^2 e^2 + \frac{1}{2} \delta^T \beta \beta^T \delta - \varepsilon_\delta \delta^T \delta, \tag{60}
\]

\[
\leq - \frac{1}{2} \dot{x}^T \dot{x} - \left( c - \frac{1}{2} \right) e^2 - \frac{1}{2} \varepsilon_\delta \delta^T \delta. \tag{61}
\]
Using (61), we conclude
\[ V(t) \leq V(0). \] (62)

Defining
\[ \Theta(t) = [x^T(t), e(t), \tilde{\theta}^T(t), \tilde{\beta}^T(t), \delta^T(t)]^T, \] (63)

and using (56) and (62), we get
\[ |\Theta(t)|^2 \leq M_1 |\Theta(0)|^2, \] (64)

for some \( M_1 > 0 \). Taking derivative of (18) and using (19) and (35) we get
\[ \dot{\tilde{\xi}}(t) = G\tilde{\xi}(t). \] (65)

Since \( G \) is Hurwitz, using (65), we have
\[ |\tilde{\xi}(t)| \leq M_2 e^{-\alpha_1 t} |\tilde{\xi}(0)|, \] (66)

for some \( M_2, \alpha_1 > 0 \). By using (66), we write
\[ |\tilde{\xi}(t)| \leq M_2 |\tilde{\xi}(0)| + M_3 |\Theta(0)|, \] (67)

for some \( M_3 > 0 \). By using (64) and (67), we obtain
\[ |\Xi(t)| \leq M_4 |\Xi(0)|, \] (68)

where
\[ \Xi(t) = \begin{bmatrix} \Theta(t) \\ \tilde{\xi}(t) \end{bmatrix}, \] (69)

for some \( M_4 > 0 \). This proves part (a) of Theorem 1.

For all \( \Xi \), the right-hand side of (51) and (52) are continuous in \( \Xi \) and \( t \), which implies that the right-hand side of (61) is continuous in \( \Xi \) and \( t \). Furthermore, the right-hand side of (61) is zero at \( \Xi = 0 \). By the LaSalle-Yoshizawa theorem, (61) ensures that \( \dot{\xi}, e \) and \( \dot{\delta} \) converge to zero as \( t \to \infty \).

We represent the closed-loop of \((x, \theta)\) system as a linear time-varying (LTV) system which is given by
\[ \dot{\zeta} = E(t)\zeta + F(t)d, \] (70)

where
\[ E(t) = \begin{bmatrix} A_{cl} & B_{\xi}^T \\ \gamma_1 \xi B_{\delta}^T & PA_{cl} & \gamma_1 \xi B_{\delta}^T \end{bmatrix}, \]
\[ F(t) = \begin{bmatrix} B_{\theta}^T \\ \gamma_2 \xi B_{\delta}^T \end{bmatrix}, \]
\[ \zeta = \begin{bmatrix} x^T, \theta^T \end{bmatrix}^T, \]
\[ d = \begin{bmatrix} \delta^T, e \end{bmatrix}^T. \]

We first show that the equilibrium \( \zeta = 0 \) of the homogenous part of the LTV system (70) is exponentially stable. Towards that end, we choose the following Lyapunov function
\[ V_c = \frac{1}{2} \xi^TP_c\xi, \] (75)

where
\[ P_c = \begin{bmatrix} P & 0 \\ 0 & \frac{1}{\gamma}I \end{bmatrix}. \] (76)

Taking the derivative of \( V_c \), we get
\[ \dot{V}_c = \frac{1}{2} \xi^T \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \xi, \] (77)

where
\[ Q_1 = A_{cl}^T P + PA_{cl}, \]
\[ Q_2 = (A_{cl}^T P A_{cl}^{-1} + P) B_{\xi}^T, \]
\[ Q_3 = \xi B_{\delta}^T (A_{cl}^T P + PA_{cl}^{-1}) B_{\xi}^T. \]

By pre and post multiplying (9) by \( A_{cl}^T \) and \( A_{cl} \) and using the fact that \( A_{cl} = (A^{-1} + A^{-1}BK)^{-1} \), we obtain
\[ A_{cl}^T P + PA_{cl} = -2A_{cl}^T A_{cl}. \] (78)

Pre-multiplying (9) by \( A_{cl}^T \), we get
\[ A_{cl}^T P A_{cl}^{-1} + P = -2A_{cl}^T. \] (79)

Post-multiplying (9) by \( A_{cl} \), we get
\[ P + A_{cl}^T PA_{cl} = -2A_{cl}. \] (80)
Substituting (9), (78)–(80) into (77), we get

\[ \dot{V} = -\zeta^T \left[ A_{cl}^T \xi + A_{cl}^T B \xi^T \right] \zeta. \]  

(81)

Defining

\[ C^T(t) = \begin{bmatrix} A_{cl} & B \xi^T \end{bmatrix}, \]  

(82)

we get

\[ \dot{V} = \frac{1}{2} \zeta^T (E^T(t)P_e + P_e E(t)) \zeta = -\zeta^T C(t)C(t)^T \zeta. \]  

(83)

Therefore, it follows that \( P_e \), as defined in (76), satisfies the following inequality

\[ E^T(t)P_e + P_e E(t) + \alpha C^T(t)C(t) \leq 0 \]  

(84)

for some \( \alpha > 0 \).

The equilibrium \( \zeta = 0 \) of the homogenous part of (70) is exponentially stable if \((C(t), E(t))\) is a uniformly completely observable (UCO) pair [36]. For a bounded \( H(t) \), the pairs \((C(t), E(t))\) and \((C(t), E(t) + H(t)C(t)^T)\) have the same UCO property [36]. Choosing

\[ H(t) = \begin{bmatrix} \frac{I}{\zeta^T P} \\ \frac{-1}{\zeta^T P} \end{bmatrix}, \]  

(85)

we write the system corresponding to the pair \((C, E + HC^T)\) as

\[
\begin{align*}
\dot{Y} &= 0 \\
y &= C^T(t)Y.
\end{align*}
\]  

(86-87)

The state transition matrix of (86) is \( \Phi = I \). Therefore, \((C, E + HC^T)\) is a UCO pair if there exist positive constants \( \alpha_2, \alpha_3, \rho \) such that the observability Gramian satisfies

\[ \alpha_2 I \geq \int_{t_0}^{t_0 + \rho} C(t)C^T(t) dt \geq \alpha_3 I, \]  

(88)

for all \( t_0 \geq 0 \). Since \( \zeta \) is bounded, recalling (82), the upper bound of (88) is satisfied. We now prove the lower bound in (88). Calculating the integral in (88), we get

\[
X = \int_{t_0}^{t_0 + \rho} C(t)C^T(t) dt = \begin{bmatrix} A_{cl}^T \rho & A_{cl}^T B \int_{t_0}^{t_0 + \rho} \xi^T dt \\ A_{cl}^T B \int_{t_0}^{t_0 + \rho} \xi^T dt & \int_{t_0}^{t_0 + \rho} \xi^T B \xi^T dt \end{bmatrix}. \]  

(89)

Let \( S_h \) be the Schur complement of \( A_{cl}^T A_{cl} \rho \) in \( X \), where

\[
S_h = \int_{t_0}^{t_0 + \rho} \xi^T B \xi^T dt - \frac{1}{\rho} \int_{t_0}^{t_0 + \rho} \xi^T dt \int_{t_0}^{t_0 + \rho} \xi^T dt. \]  

(90)

Since \( A_{cl}^T A_{cl} \rho \) is positive definite, \( X \) is positive definite if and only if \( S_h \) is positive definite. Since \( B^T B \) is a positive scalar, according to Lemma 3 there exists a positive \( \rho \) such that for all \( t_0 > 0, S_h > 0 \). Hence, \((C, E + HC^T)\) is UCO, which implies that \((C, E)\) is UCO. Therefore, the state transition matrix \( \Phi(t, t_0) \) corresponding to \( E(t) \) in (70) satisfies

\[
\| \Phi(t, t_0) \| \leq \kappa_0 e^{-\gamma_h (t-t_0)} \]  

(91)

for some positive constants \( \kappa_0, \gamma_h \). From (74), \( d(t) \) is bounded and, from (72), \( F(t) \) is bounded. Recalling that it has already been established that \( d(t) \) goes to zero, from (70) and (91) it follows that \( \zeta \) is bounded and \( \zeta^T = [x^T, \theta^T]^T \rightarrow 0 \) as \( t \rightarrow \infty \). By using (65) and the fact that \( G \) is Hurwitz, we conclude that \( \tilde{e}(t) \) converges to zero as \( t \rightarrow \infty \). Furthermore, thanks to Lemma 2, \( \tilde{e}^T(t) e(t) \rightarrow 0 \) as \( t \rightarrow \infty \). This proves part (b) of Theorem 1.

4 Simulation Results

We illustrate the performance of our controller with a third-order system with \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, a^T = \begin{bmatrix} 1 & 2 \end{bmatrix}, b_u = b_p = 1, \) the unknown disturbance \( \nu(t) = 1.2 \sin(0.8 \pi t + \pi/4) - 0.5 \sin(t + \pi/2) \) and initial conditions \( x(0) = [1 \ -2.5 \ 0]^T, p(0) = 0.5 \). The control gain \( K \) is chosen such that the eigenvalues
of $A_{cl}$ are $-3, -4$ and $c = 0.8$. For the update law, we choose $\gamma = \gamma_0 = 2$. Finally, the controllable pair $(G, l)$ is chosen as $G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.37 & -12.12 & -12.60 & -5.80 \end{bmatrix}$, $l = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. From Figures 1 and 2, one can observe that $x(t)$ converges to zero and the unknown disturbance is perfectly estimated, as Theorem 1 predicts.

5 Conclusions
In the present work we design an adaptive backstepping controller by using state derivative of the main system and state of the input subsystem to cancel unmatched unknown sinusoidal disturbances forcing a linear time-invariant systems. We prove that the equilibrium of the closed loop adaptive system is stable and the state of the considered error system goes to zero as $t \to \infty$ with perfect disturbance estimation. The effectiveness of our controller is demonstrated with a numerical example.

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REFERENCES


