ABSTRACT

In this paper, an adaptive observer and backstepping controller are designed to cancel and estimate sinusoidal disturbances forcing a linear time-invariant by using only the measurements of the state-derivatives. The parametrization of the sinusoidal disturbance as the output of a known feedback system with an unknown output vector that depends on unknown disturbance parameters with the necessary filter designs enables to approach the problem as an adaptive control problem. An observer is designed for the unmeasured virtual input to apply a backstepping procedure which handles the unmatched disturbance and input condition. Firstly, it is shown that the disturbance and the unmeasured actuator state are observed perfectly in the open loop case. Secondly, the closed loop case is considered and it is proven that the equilibrium of the closed-loop adaptive system is stable and the state of the considered original system converge to zero as \( t \to \infty \) with perfect disturbance estimation. The effectiveness of the controller and the observers are illustrated with a simulation example of a third order system.

1 Introduction

The problem of canceling sinusoidal disturbances in dynamical systems is a fundamental control problem, with many applications such as vibrating structures [1], active noise control [2] and rotating mechanisms control [3]. The common method to approach this problem is the internal model principle for which a general solution is given in [22], [23] in the case of linear systems. In the internal model approach, the disturbance is modeled as the output of a linear dynamic system which is called an exosystem. Then the effect of the disturbance on the plant response can be completely compensated by adding a replica of the exosystem model in the feedback loop.

The output regulation problem for minimum phase, uncertain nonlinear systems is solved in [5], [8], and extended for nonminimum phase plants in [7]. Moreover, the designs for nonlinear systems are proposed in [10], [11], [19], [21]. The solutions of disturbance cancelation and output regulation also exist for continuous-time LTI systems [9], [6], [12], [13], [14], [15], [20] and discrete-time LTI systems [16]. In all of these references, the controllers are designed by using the measurement of the state or an output.

In the last decade, the state derivative feedback control has been considered by many researchers [24]– [28] due to its various advantages in applications. In most practical systems, using accelerometers as sensors is easier, cheaper and more reliable than using position sensors.

In this note, an adaptive observer and controller are designed to estimate and cancel the unmatched unknown sinusoidal disturbances forcing general LTI systems by using measurement of the state-derivative of the system. It is proven that the equilibrium of the closed loop system is stable and the state converges to zero as \( t \to \infty \) with perfect disturbance estimation. The solution for the case where the disturbance is required to match the input is given in [29].

In Section 2, the problem statement is given. In Section 3,
the observer designs for unmeasured virtual input and unknown disturbance are presented with the proof of the convergence. In Section 4, the steps of the design are carried out and the adaptive controller is given with the stability theorem. The proof of the stability theorem is given in Section 5. A simulation example is presented in Section 6.

2 Problem Statement

Consider the single-input LTI system

\[
\begin{align*}
\dot{x} &= Ax + B(p + v), \\
\dot{p} &= a_p x + a_p p + bu,
\end{align*}
\]

with the states \( x \in \mathbb{R}^n \) and \( p \in \mathbb{R} \), input \( u \in \mathbb{R} \) and sinusoidal disturbance \( v \in \mathbb{R} \) given by

\[
v(t) = \sum_{i=1}^{q} g_i \sin(\omega_i t + \phi_i),
\]

where \( i \neq j \Rightarrow \omega_i \neq \omega_j, \omega_i \in \mathbb{Q}_+, g_i, \phi_i \in \mathbb{R} \). In this problem, the disturbance \( v(t) \) and the control input \( u(t) \) are not matched (i.e., they are unmatched) because they do not appear in the same equation. The state \( p(t) \) is considered as the virtual input of system (1). The states \( x(t), p(t) \) and the disturbance \( v(t) \) are not measured but \( \dot{x}(t) \) and \( \dot{p}(t) \) are measured.

The sinusoidal disturbance \( v \) can be represented as the output of a linear exosystem,

\[
\begin{align*}
\dot{w} &= Sw \quad (4) \\
v &= h^T w \quad (5)
\end{align*}
\]

where \( w \in \mathbb{R}^{2q} \). The matrix \( S \) depends on the unknown and rational distinct frequencies of the sinusoidal disturbance \( v \), while the uncertainty of amplitude and phase is related to the unknown initial condition of (4). Since \( v(t) \) is unknown, \( S \) is also unknown but without loss of generality, the output vector \( h^T \) can be chosen (and is thus known) with the property that the pair \((h^T, S)\) is observable.

The following assumptions are made regarding the plant (1)–(2) and the exosystem (4)–(5):

**Assumption 1.** The system matrix \( A \) is invertible.

**Assumption 2.** The pair \((A,B)\) is controllable.

**Assumption 3.** The input coefficient \( b \neq 0 \) and \( a_p - a_p^T A^{-1} B \neq 0 \).

**Assumption 4.** The number of distinct frequencies, \( q \), is known.

**Assumption 5.** \( g_i \neq 0 \) for all \( i \in \{1, \ldots, q\} \) (i.e., \( w(0) \neq 0 \)).

3 State and Disturbance Observer Design

Firstly, the disturbance and its derivative are represented as a constant unknown vector multiplied by a known regressor, plus an unknown exponentially decaying disturbance. In the second part, based on the obtained representation, an observer is designed for the virtual input \( p(t) \).

3.1 Disturbance Representation

The disturbance is parameterized by following [17]. Let \( G \in \mathbb{R}^{2q \times 2q} \) be a Hurwitz matrix with distinct eigenvalues and let \((G,l)\) be a controllable pair. Since \((h^T, S)\) is observable and the spectra of \( S \) and \( G \) are disjoint the unique solution \( M \in \mathbb{R}^{2q \times 2q} \) of the Sylvester equation

\[
MS - GM = lh^T.
\]

is invertible [32]. The change of coordinates \( z = Mw \) transform the exosystem (4)–(5) into the form

\[
\begin{align*}
\dot{z} &= Gz + lv, \\
v &= \theta^T \dot{z},
\end{align*}
\]

where

\[
\theta^T = h^T (MS)^{-1}.
\]

Differentiating (8) and substituting \( \dot{z} = G\dot{z} + l\dot{v} \), we obtain

\[
\dot{v} = \beta^T \dot{z},
\]

where

\[
\beta^T = \frac{1}{1 - \theta^T l} \theta^T G.
\]

Post-multiplying (6) by \((MS)^{-1}\) and using (9), the equation matrix \( I - lG^T = GMS^{-1}M^{-1} \) is obtained. Using the Sylvester’s determinant theorem [31], the fact that \( \det M^{-1} = \frac{1}{\det S} \) and noting that \( G \) and \( S \) have \( 2q \) poles on the left half plane and on the imaginary axis, respectively, it is shown that \( 1 - \theta^T l > 0 \).

The unknown external disturbance \( v \) and \( \dot{v} \) are represented as the product of an unknown constant and the vector \( \dot{z} \) in (8) and (10), respectively. However, \( \dot{z} \) can not be used in a control design, since it can not be measured. To overcome this problem, a conceptual observer is designed. The following lemma establishes the properties of the observer.
Lemma 1. The inaccessible disturbance $v$ and $\dot{v}$ can be represented in the form

\begin{align}
v &= \theta^T \xi + \theta^T \delta, \quad (12) \\
\dot{v} &= \beta^T \xi + \beta^T \delta, \quad (13)
\end{align}

where

\begin{align}
\xi &= \eta + N \dot{x}, \quad (14) \\
\dot{\eta} &= G(\eta + N \dot{x}) - N(A \dot{x} + B \dot{p}), \quad (15)
\end{align}

with $N \in \mathbb{R}^{2q \times n}$ which is given by

\begin{equation}
N = \frac{1}{B^T B} \mathbb{1} B^T, \quad (16)
\end{equation}

where the given $N$ is one of the many solutions of the equation

\begin{equation}
NB = \mathbb{1}, \quad (17)
\end{equation}

and the estimation error $\delta \in \mathbb{R}^{2q}$ obeys the equation

\begin{equation}
\dot{\delta} = G \delta. \quad (18)
\end{equation}

Proof. Define an estimation error

\begin{equation}
\delta = \hat{z} - \hat{\xi}. \quad (19)
\end{equation}

The equation (18) is obtained by differentiating $\delta$ with respect to time and using the time derivative of (1), (7) and (17). Substitution of (19) into (8) and (10) yields (12) and (13), respectively.

3.2 Observer Design

The unmeasured state $p$ is represented as

\begin{equation}
p = \hat{p} + \frac{1}{a_p} \left( (1 - c_e_p \hat{p}) e_p + a_e^T \overline{B}(\theta_T \xi + \theta^T \delta) \right), \quad (20)
\end{equation}

where

\begin{align}
\dot{\hat{p}}(t) &= \frac{\pi_p}{1 - c_e_p} \hat{p}(t) + \frac{1}{1 - c_e_p} (a_e^T A^{-1} \dot{x}(t) - a_e^T \overline{B} \theta_T \xi(t) \\
&+ b \nu(t) - c_e_p \hat{p}) \quad (21) \\
\dot{e}_p &= \hat{p} - \hat{\dot{p}}, \quad (22)
\end{align}

with

\begin{align}
\overline{B} &= A^{-1} B, \quad (23) \\
\pi_p &= \alpha_p - a_e^T \overline{B}, \quad (24)
\end{align}

\begin{equation}
\text{sign}(\pi_p)c_{e_p} > \text{sign}(\alpha_p) + \frac{|\alpha_p|}{2} \quad \text{and} \quad \theta = \hat{\theta} - \hat{\dot{\theta}} \quad \text{where}
\end{equation}

\begin{equation}
\hat{\dot{\theta}} = \kappa_\theta \frac{1}{a_p} \xi(t) a_e^T \overline{B} \dot{e}_p(t). \quad (25)
\end{equation}

Since $\dot{p}$ is measured, $\dot{e}_p$ is also measured and implementable.

In this section, we present the result for the open loop state and disturbance estimation. The following lemma is used in the proof of theorems.

Lemma 2. There exists $\rho > 0$ such that for all $t_0 \geq 0$, the following holds

\begin{equation}
Q_p(\rho, t_0) = \int_{t_0}^{\rho + t_0} \xi(t) \xi^T(t) dt \\
+ \frac{1}{\rho} \int_{t_0}^{\rho + t_0} \dot{\xi}(t) dt \int_{t_0}^{\rho + t_0} \dot{\xi}(t) dt > 0. \quad (26)
\end{equation}

Proof. By differentiating (19) with respect to time and using the fact that $\dot{\xi} = G \dot{\xi} + \hat{v}$ and (18), we obtain

\begin{equation}
\dot{\xi} = G \xi + \hat{v}, \quad (27)
\end{equation}

where $\hat{v}(t) = \sum_{i=1}^{\nu} \theta_i \theta_i \cos(\omega_i t + \phi_i)$. $G$ is Hurwitz and the pair $(G, \mathbb{1})$ is controllable. The proof for the signal given by (27) is given in [29].

The result of the section is given by the following theorem.

Theorem 1. Consider the disturbance observer (14), (15), the state observer (21) and the update law (25), under Assumptions 1–4, for all initial conditions $x(0) \in \mathbb{R}^n$, $\tilde{\theta}(0) \in \mathbb{R}^{2q}$, $e_p(0) \in \mathbb{R}$ and all $w(0) \in \mathbb{R}^{2q}$ such that Assumption 5 holds, the signals $e_p(t), \tilde{\theta}(t), \hat{\delta}(t), \hat{v}(t) - \hat{\dot{\theta}}(t) \dot{\xi}(t)$ converge to zero as $t \to \infty$.

Proof. We represent the closed-loop of $(e_p, \tilde{\theta})$ system as a linear time-varying (LTV) system which is given by

\begin{equation}
\dot{\xi}(t) = E(t) \xi(t) + F(t) \delta(t), \quad (28)
\end{equation}
where

\[
E(t) = \begin{bmatrix}
\frac{\pi_p}{1-c_{ep}} & -\frac{1}{1-c_{ep}}a_x^T B \xi(t) \\
-\frac{\pi_p}{1-c_{ep}}a_x^T B \xi(t) & \frac{\pi_p}{1-c_{ep}}(a_x^T B)^2 \xi(t) \xi(t)^T
\end{bmatrix},
\]

\[
F(t) = \begin{bmatrix}
-\frac{\pi_p}{1-c_{ep}}a_x^T B \hat{\theta}^T \\
\frac{\pi_p}{1-c_{ep}}(a_x^T B)^2 \hat{\theta}^T
\end{bmatrix},
\]

\[
\dot{\zeta}(t) = \begin{bmatrix}
e_p(t), \hat{\theta}^T(t)
\end{bmatrix}^T,
\]

The main aim is to show the convergence of \(\zeta(t)\). Firstly, we show the exponential stability of the equilibrium \(\zeta(t) = 0\) of the homogeneous part of the LTV system (28). To this end, the Lyapunov function is chosen as follows

\[
V_{LTV} = \frac{1}{2} \dot{\zeta}(t)^T P_{LTV} \zeta(t),
\]

where

\[
P_{LTV} = \begin{bmatrix}
1 & 0 \\
0 & \frac{1}{\pi_p}
\end{bmatrix}.
\]

Taking the time derivative of \(V_{LTV}\), we get

\[
\dot{V}_{LTV} = \dot{\zeta}(t)^T \begin{bmatrix}
\frac{\pi_p}{1-c_{ep}} & -\frac{1}{1-c_{ep}}a_x^T B \xi(t) \\
-\frac{1}{1-c_{ep}}a_x^T B \xi(t) & \frac{\pi_p}{1-c_{ep}}(a_x^T B)^2 \xi(t) \xi(t)^T
\end{bmatrix} \zeta(t).
\]

Defining

\[
C(t)^T = \begin{bmatrix}
\sqrt{\frac{\pi_p}{1-c_{ep}}} & -\frac{1}{\sqrt{\pi_p}(1-c_{ep})}a_x^T B \xi(t)^T
\end{bmatrix},
\]

we get

\[
\dot{V}_{LTV} = \frac{1}{2} \dot{\zeta}(t)^T (E(t)^T P_{LTV} + P_{LTV} E(t)) \zeta(t)
\]

\[
= -\zeta(t)^T C(t) C(t)^T \zeta(t).
\]

where \(-\frac{\pi_p}{1-c_{ep}} > 0\) due to the fact that \(\text{sign}(\alpha_p)c_{ep} > \text{sign}(\alpha_p)\).

Therefore, it follows that \(P_{LTV}\) satisfies the following inequality

\[
E(t)^T P_{LTV} + P_{LTV} E(t) + \alpha C(t)^T C(t) \leq 0,
\]

for some \(\alpha > 0\).

The equilibrium of the homogenous of (28) is exponentially stable if the pair \((C(t), E(t))\) is uniformly completely observable (UCO) [36]. For a bounded \(H(t)\), the pairs \((C(t), E(t))\) and \((C(t), E(t) + H(t) C(t)^T)\) have the same (UCO) property [36]. Choosing

\[
H(t) = \begin{bmatrix}
\sqrt{-\frac{\pi_p}{1-c_{ep}}} \\
-\frac{\pi_p}{1-c_{ep}}a_x^T B \xi(t)
\end{bmatrix},
\]

the system corresponding to the pair \((C(t), E(t) + H(t) C(t)^T)\) is written as

\[
\dot{y}(t) = 0,
\]

\[
y(t) = C(t)^T y(t)
\]

The state transition matrix of (39) is \(\Phi(t) = I\). Therefore, \((C(t), E(t) + H(t) C(t)^T)\) is a UCO pair if there exists positive constants \(\alpha_2, \alpha_3, \rho\), such that the observability gramian satisfies

\[
\alpha_2 t \geq \int_{t_0}^{t_0 + \rho} C(t) C(t)^T dt \geq \alpha_3 t,
\]

for all \(t_0 \geq 0\). Since \(\xi(t)\) is bounded, recalling (34), the upper bound of (41) is satisfied. The lower bound in (41) is now proven. Calculating the integral in (41), we get

\[
X = \int_{t_0}^{t_0 + \rho} C(t) C(t)^T dt
\]

\[
= \begin{bmatrix}
\frac{-\pi_p}{1-c_{ep}} & \frac{1}{1-c_{ep}}a_x^T B \int_{t_0}^{t_0 + \rho} \xi(t)^T dt \\
\frac{1}{1-c_{ep}}a_x^T B \int_{t_0}^{t_0 + \rho} \xi(t)^T dt & \frac{1}{\pi_p(1-c_{ep})} \int_{t_0}^{t_0 + \rho} \xi(t)^T \xi(t)^T dt
\end{bmatrix}
\]

Let \(S_h\) be the Schur complement of \(\rho\) in \(X\), where

\[
S_h = -\frac{(a_x^T B)^2}{\pi_p(1-c_{ep})} \left( \int_{t_0}^{t_0 + \rho} \xi(t)^T dt - \frac{1}{\rho} \int_{t_0}^{t_0 + \rho} \xi(t) dt \int_{t_0}^{t_0 + \rho} \xi(t)^T dt \right).
\]

Since \(\rho\) and \(-\frac{\pi_p}{1-c_{ep}}\) are positive definite if and only if \(S_h\) is positive definite. Since \(-\frac{(a_x^T B)^2}{\pi_p(1-c_{ep})}\) is a positive scalar, according to Lemma 2 there exists a positive \(\rho\) such that for all
$t_0 > 0$, $S_h > 0$. Hence, $(C,E + HC^T)$ is UCO, which implies that $(C,E)$ is UCO. Therefore, the state transition matrix $\Phi(t,t_0)$ corresponding to $E(t)$ in (28) satisfies

$$\| \Phi(t,t_0) \| \leq \kappa_0 e^{-\gamma_0(t-t_0)}$$

(44)

for some positive constants $\kappa_0, \gamma_0$. Since $G$ is Hurwitz, we have that

$$|\delta(t)| = |e^{G(t-h_0)} \delta(0)| \leq \kappa_1 e^{-\gamma_1(t-h_0)} |\delta(0)|$$

(45)

for some positive constants $\kappa_1, \gamma_1$. The solution of (28) is written as

$$\zeta(t) = \Phi(t,0) \zeta(0) + \int_0^t \Phi(t,\tau) F(\tau) \delta(\tau) d\tau.$$  

(46)

Using the fact that $\bar{\xi}(t)$ is bounded, recalling (27), from (30), $F(t)$ is bounded. Using (44)–(46), we get

$$|\zeta(t)| \leq \kappa_0 e^{-\gamma_0 t} |\zeta(0)| + \kappa_1 \kappa_0 \sup_{0 \leq \tau \leq t} |F(\tau)| (\gamma_1 - \gamma_0 \min\{|\gamma_1, \gamma_0\}|).$$

(47)

From (47), it is concluded that $\zeta(t) = [e_p(t), \tilde{\theta}^T(t)]^T$ converge to zero as $t \to \infty$. From Lemma 1, using $\delta(t)$ and $\tilde{\theta}^T(t)$ converge to zero, it is concluded that $v(t) - \tilde{\theta}^T(t) \bar{\xi}(t)$.

The result given in Theorem 1 is beneficial in order to estimate the unknown sinusoidal disturbance $v(t)$ and the state $p(t)$ in open loop. In the following section, the adaptive control design for the input $u(t)$ and the stability condition of the closed loop system is given.

4 Control Design and Stability

In order to design a controller for the actual input $u$, a backstepping procedure is applied by designing a control law for the virtual input $\hat{p}$. However, $p$ is not measured. This problem is handled by using (20).

4.1 Backstepping Design

The problem is reformulated as an adaptive control problem and an adaptive backstepping controller is developed by representing the system in the reciprocal state space (RSS) form which depends on switching the state vector with its derivative [28]. The reciprocal state space representation (RSS) is a beneficial platform for a state derivative feedback control design. Substituting (12) and (20) into (1), the RSS form of system (1) is written as

$$x = A^{-1} x - B \left( \hat{p} + \frac{1}{\bar{a}_p} \left( (1 - c_{e_p}) \hat{e}_p + a_{\tilde{\theta}}^T \tilde{\theta}^T \xi \right) + \theta^T \xi + \frac{a_{\tilde{\theta}}^T \theta^T}{\bar{a}_p} \bar{\delta} \right).$$

(48)

The signal $\hat{p}$ is obtained by using (21) that contains only measured terms and the control input, $u$. Therefore, a backstepping procedure is applied by considering $\hat{p}$ as the virtual controller.

The backstepping procedure has three main steps. Firstly, a control law is designed for the virtual input $\hat{p}$. The desired value of $\hat{p}$ is given by

$$\hat{p}_d(t) = -K \dot{x} - \hat{\theta}^T \xi - \frac{1 - c_{e_p}}{\bar{a}_p} \hat{e}_p$$

(49)

where the control gain $K \in \mathbb{R}^{1 \times n}$ is chosen so that $(A^{-1} + A^{-1}BK)$ is Hurwitz with the positive definite matrix $P$ which is the solution of the matrix equation

$$(A^{-1} + A^{-1}BK)^T P + P(A^{-1} + A^{-1}BK) = -4I.$$  

(50)

Under Assumptions 1 and 2, there exists a control gain $K \in \mathbb{R}^{1 \times n}$ such that $(A^{-1} + A^{-1}BK)$ is Hurwitz [24]. Secondly, an error term is defined to represent the difference between actual $\hat{p}$ and its desired value $\hat{p}_d$. The error term is given by

$$e_d(t) = \hat{p}(t) - \hat{p}_d(t).$$

(51)

The system $(x, p)$ is converted to the system $(x, e_p, e_d)$ which is
given by
\begin{align*}
    x &= A_{cl}^{-1} \dot{x} - B \left( e_d + \frac{a_p}{\alpha_p} \left( \theta^T \xi + \theta^T \tilde{\delta} \right) \right), \\
    \dot{e}_p &= -\frac{\alpha_p}{1 - c_{ep}} e_p - \frac{a_p}{\alpha_p} \left( \theta^T \xi + \theta^T \tilde{\delta} \right), \\
    \dot{e}_d &= -\frac{\alpha_p}{1 - c_{ep}} \dot{p}(t) + \frac{1}{1 - c_{ep}} \left( a_p^T A^{-1} \dot{x}(t) - a_p^T B \theta^T \xi(t) \right) + b u(t) - c_{ep} \dot{p} - \left( - \left( K + \frac{a_p}{\alpha_p} \theta^T N \right) (A \dot{x} + B \dot{p}) - \frac{a_p}{\alpha_p} \left( \theta^T \xi + \theta^T \tilde{\eta} \right) - \dot{e}_p \\
    &+ \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right) \hat{\beta}^T \xi \right) - \dot{x}^T PB \left( \frac{1}{2} \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right)^2 + c_{ed} \right) \left( e_d \right) \right) .
\end{align*}
where
\begin{equation}
    A_{cl} = (A^{-1} + A^{-1} BK)^{-1}
\end{equation}

4.2 Main Controller and Stability Statement

The adaptive controller for system (1), (2), (4), (5) is given by
\begin{equation*}
    u = -\frac{1}{\beta} \left( a_p^T A^{-1} \dot{x}(t) - a_p^T B \theta^T \xi(t) - c_{ep} \dot{p} \right) + \alpha_p \dot{p}(t) + \left( (1 - c_{ep}) \left( - \left( K + \frac{a_p}{\alpha_p} \theta^T N \right) (A \dot{x} + B \dot{p}) - \frac{a_p}{\alpha_p} \left( \theta^T \xi + \theta^T \tilde{\eta} \right) - \dot{e}_p \\
    &+ \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right) \hat{\beta}^T \xi \right) - \dot{x}^T PB \left( \frac{1}{2} \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right)^2 + c_{ed} \right) \left( e_d \right) \right).
\end{equation*}

where $c_{ed} > 0$. The update laws are given by
\begin{align*}
    \dot{\hat{\theta}} &= \kappa_\theta \frac{1}{\alpha_p} \xi(t) \left( a_p^T B e_p - a_p x^T P B \right), \\
    \dot{\hat{\beta}} &= -\kappa_\beta \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right) \xi e_d
\end{align*}

Remark 1. The update law of $\dot{\hat{\theta}}(t)$ given by (57) has an additional term, $-\kappa_\theta \xi(t) \frac{a_p}{\alpha_p} \dot{x}(t)^T P B$, when it is compared with the update law given by (25) for the open loop estimation. The additional term comes out due to the backstepping procedure which is needed to design an adaptive control law for the system input, $u(t)$. For open loop estimation, update law (25) is used in order to estimate $\theta$. However, for control purpose, in order to achieve stability and convergence, update law (57) is employed. The details are given in the proofs of each theorem.

In order to state the stability theorem, define the signal
\begin{equation}
    \bar{\xi} = \xi - \overline{\xi},
\end{equation}
where $\overline{\xi} = \frac{1}{t} \int_0^t e^{G(t-\tau)} G \nu(\tau) d\tau$.

Theorem 2. Consider the closed-loop system consisting of the plant (1), (2) forced by the unknown sinusoidal disturbance (4), (5), the disturbance observer (14), (15), the state observer (21) and the adaptive controller (56)–(58). Under Assumptions 1–4, the followings hold:

(a) The equilibrium $x = 0, \theta = \hat{\theta} = \delta = \overline{\xi}, e_p = e_d = 0$ is stable,
(b) For all initial conditions $x(0) \in \mathbb{R}^n, \overline{\xi}(0) \in \mathbb{R}^{2q}, \hat{\theta}(0) \in \mathbb{R}^{2q}, \overline{\theta}(0) \in \mathbb{R}^{2q}, e_p(0) \in \mathbb{R}, e_d(0) \in \mathbb{R}$ and all $w(t) \in \mathbb{R}^{2q}$ such that Assumption 5 holds, the signals $x(t), e_p(t), e_d(t), \overline{\theta}, \delta, \overline{\xi}, \nu(t) - \theta^T \xi$ converge to zero as $t \to \infty$.

5 Stability Proof

In this section, the proof Theorem 2 is given.

Proof of Theorem 2: The closed loop system is written as
\begin{align*}
    \dot{x} &= A_{cl} x + B \left( e_d + \frac{a_p}{\alpha_p} \left( \theta^T \xi + \theta^T \tilde{\delta} \right) \right), \\
    \dot{e}_p &= -\frac{\alpha_p}{1 - c_{ep}} e_p - \frac{a_p}{\alpha_p} \left( \theta^T \xi + \theta^T \tilde{\delta} \right), \\
    \dot{e}_d &= -\left( c_{ed} + \frac{1}{2} \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right)^2 \right) e_d \\
    &- \left( \frac{a_p^T B}{\alpha_p} - KB - \frac{a_p}{\alpha_p} \theta^T I \right) \left( \beta^T \xi + \beta^T \tilde{\delta} \right),
\end{align*}
where
\begin{equation}
    B = A_{cl} B.
\end{equation}
The stability of the equilibrium of the closed loop system is established with the use of the following Lyapunov function,

$$ V = \frac{1}{2} \left( x^T P x + e_p^2 + e_d^2 + \bar{\theta}^T \bar{\theta} + \delta^T P \delta \right) $$  \hspace{1cm} (64) 

where

$$ G^T P \delta + P \delta G = -2I, $$

$$ e_\delta = e + \frac{1}{2} \left( \frac{a_p}{\bar{\alpha}_p} \right)^2 \lambda_{\max} \left( \theta \bar{B}^T P \bar{B} \theta^T \right) $$

$$ + \frac{1}{2} \left( \frac{a_T^2 \bar{B}}{\bar{\alpha}_p} \right)^2 \lambda_{\max} \left( \theta \theta^T \right) + \frac{1}{2} \lambda_{\max} \left( \beta \beta^T \right). $$  \hspace{1cm} (66) 

Taking time derivative of $V$, in view of (13),(57),(58) and (60)--(62), we obtain

$$ \dot{V} = -2x^T \dot{x} + \left( 1 - \frac{ce_p}{\bar{\alpha}_p} \right) e_p^2 - \left( \frac{a_T^2 \bar{B}}{\bar{\alpha}_p} - KB - \frac{a_p}{\bar{\alpha}_p} \dot{\bar{\theta}}^T \right)^2 $$

$$ + c_\delta e_d^2 - e_\delta^2 \delta^T \delta - x^T P \bar{B} \left( e_d + \frac{a_p}{\bar{\alpha}_p} \theta^T \dot{\theta} \right) $$

$$ + \left( KB + \frac{1}{\bar{\alpha}_p} \left( a_p \dot{\bar{\theta}}^T l - a_T^2 \bar{B} \right) \right) \beta^T \delta e_d. $$  \hspace{1cm} (67) 

Using Young’s inequality for the cross terms, we get

$$ \dot{V} \leq -x^T \dot{x} + \left( 1 - \frac{ce_p}{\bar{\alpha}_p} + \frac{1}{2} \right) e_p^2 - c_\delta e_d^2 - c_\delta^2 \delta^T \delta. $$  \hspace{1cm} (68) 

Using the fact that $\left( 1 - \frac{ce_p}{\bar{\alpha}_p} + \frac{1}{2} \right) < 0$ due to $\text{sign}(\bar{\alpha}_p)ce_p > \text{sign}(\bar{\alpha}_p) + \frac{|\alpha_p|}{2}$, from (68), we conclude

$$ V(t) \leq V(0). $$  \hspace{1cm} (69) 

Defining

$$ \Theta(t) = \left[ x^T(t), e_p(t), e_d(t), \bar{\theta}^T, \beta^T, \delta^T \right]^T, $$  \hspace{1cm} (70) 

and using (64) and (69), we get

$$ |\Theta(t)|^2 \leq M_1 |\Theta(0)|^2, $$  \hspace{1cm} (71) 

for some $M_1 > 0$. Taking derivative of (59) and using (27) yield

$$ \dot{\hat{\xi}}(t) = G \hat{\xi}(t). $$  \hspace{1cm} (72) 

Since $G$ is Hurwitz, using (72), we get

$$ |\hat{\xi}(t)| \leq M_2 e^{-\alpha_t |\hat{\xi}(0)|}, $$  \hspace{1cm} (73) 

for some $M_2, \alpha_1 > 0$. By using (71) and (73), we obtain

$$ |\Xi(t)| \leq M_4 |\Xi(0)| $$  \hspace{1cm} (74) 

where

$$ \Xi(t) = \left[ \Theta^T(t), \dot{\Xi}(t) \right]^T, $$  \hspace{1cm} (75) 

for some $M_4 > 0$. This proves part (a) of Theorem 2.

For all $\Xi$, the right hand side of (60)--(62) are continuous in $x$ and $t$, which implies that the right hand side of (68) is continuous in $\Xi$ and $t$. Furthermore, the right hand side of (68) is zero at $\Xi = 0$. By the LaSalle-Yoshizawa theorem, (68) ensures that $\hat{x}(t), \dot{e}_p(t), e_d(t)$ and $\delta$ converge to zero as $t \rightarrow \infty$.

The closed-loop of $(e_p, \bar{\theta})$ system as a linear time-varying (LTV) system which is given by

$$ \dot{\zeta}(t) = E(t) \zeta(t) + F_d(t) d(t), $$  \hspace{1cm} (76) 

where $E(t)$ and $\zeta(t)$ are given by (29) and (31), respectively and

$$ F_d(t) = \left[ \begin{array}{c} 0 \\ \frac{1}{\bar{\alpha}_p} \bar{\alpha}_p a_p \bar{\theta}^T P - \frac{1}{c e_p \bar{\alpha}_p} c e_p \bar{\theta}^T (a_p \bar{\theta}^T) \end{array} \right], $$

$$ d(t) = \left[ \hat{x}(t), \delta^T(t) \right]^T. $$  \hspace{1cm} (77) 

The exponential stability of the equilibrium $\zeta(t) = 0$ of the homogeneous part of the LTV system (76) is already established by (44). From (78), $d(t)$ is bounded and, from (77), $F_d(t)$ is bounded. Recalling that it has already been established that $d(t)$ goes to zero, from (76) and (44) it follows that $\zeta(t)$ is bounded and $\zeta(t) = [e_p, \bar{\theta}^T]^T \rightarrow 0$ as $t \rightarrow \infty$. By using (72) and the fact that $G$ is Hurwitz, we conclude that $\hat{\xi}(t)$ converges to zero as $t \rightarrow \infty$.

Moreover, using the fact that $x(t), \delta(t), \bar{\theta}(t)$ are bounded, recalling (69), $\hat{\xi}(t)$ is bounded, recalling (27), $\dot{\delta}(t), \bar{\theta}(t) \rightarrow \infty$ and $A_{cl}$ is Hurwitz, from (60), we conclude that $\hat{\xi}(t)$ converges to zero as $t \rightarrow \infty$. Furthermore, thanks to Lemma 1, $\hat{\theta}^T(t) \hat{\xi}(t) - \nu(t) \rightarrow 0$ as $t \rightarrow \infty$. This proves part b of Theorem 2.

\[\square\]
6 Example

We illustrate the performance of our controller with simulation examples. We consider two cases: open loop estimation and closed loop control with estimation.

6.1 Open Loop Estimation

We consider a third-order system with

\[
A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad a_t^T = [1, 2], \quad a_p = -2, \quad \text{the unknown disturbance } v(t) = \sin(2\pi t) + 0.5\sin(\frac{2\pi t}{4}) + 0.5\sin(\frac{2\pi t}{2}) \text{ and initial condition } x(0) = [-0.5 \ -0.5]^T \text{ and } p(0) = 0.5. \]

The adaptation gain \( \kappa_0 = 2000 \). Finally, the controllable pair \((G, I)\) is chosen as

\[
G = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-10.29 & -43.48 & -75.43 & -68.75 & -34.71 & -9.20
\end{bmatrix}.
\]
\[
I = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}.
\]
From Figures 1 and 2 one can observe that \( \hat{p} \) which is the estimate of \( p(t) \) converges to \( p \) and the unknown disturbance is perfectly estimated, as Theorem 1 predicts.

### 6.2 Closed Loop Control and Estimation

In this case, we consider an unstable third-order system with
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 2 \\
1 & 0 & 0
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
a^T = [1, 2], a_p = 2, \text{ the unknown disturbance } v(t) = \sin(\frac{2\pi}{5} t) + 0.5 \sin(\frac{2\pi}{3} t + \frac{\pi}{2}) \text{ and the initial conditions of the states are chosen same as the open loop case. The control gain } K \text{ is chosen such that the eigenvalues of } (A^{-1} + A^{-1}BK) \text{ are } -3 \text{ and } -2. \text{ For the update law, we choose } k_0 = 5 \text{ and } k_p = 2. \text{ Finally, the controllable pair } (G, I) \text{ is chosen as }
\[
G = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-8.58 & -20.51 & -18.11 & -7.00
\end{bmatrix}, I = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}.
\]
From Figures 3 and 4 one can observe that \( x(t) \) converges to zero and the unknown disturbance is perfectly estimated, as Theorem 2 predicts.

### 7 Conclusions

The problem of disturbance cancellation for linear systems by state-derivative feedback is considered. The problem is converted to an adaptive control problem by representing the disturbance as a constant unknown vector multiplied by a known regressor plus an exponentially decaying disturbance. Firstly, an observer is designed for the unmeasured actuator state and it is shown that the perfect open loop estimation is achieved for both the state and the unknown disturbance. Secondly, an adaptive controller is designed by employing the observer for the unmeasured state and applying a backstepping procedure. It is shown that the equilibrium of the closed loop system is stable and the state \( x(t) \) converges to zero as \( t \to \infty \) with perfect disturbance estimation. The effectiveness of the controller and observer are demonstrated with a numerical examples.

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### REFERENCES


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